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Passive magnetic shielding in static gradient fields

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The effect of passive magnetic shielding on dc magnetic field gradients imposed by both external and internal sources is studied for two idealized shield models: concentric spherical and infinitely-long cylindrical shells of linear material. It is found that higher-order multipoles of an externally applied magnetic field are always shielded progressively better for either geometry by a factor related to the order of the multipole. In regard to the design of internal coil systems, we determine reaction factors for the general multipole field and provide examples of how one can take advantage of the coupling of the coils to the innermost shell to optimize the uniformity of the field. Furthermore, we provide formulae relevant to active magnetic compensation systems which attempt to stabilize the interior fields by sensing and cancelling the exterior fields close to the outermost shell. Overall this work provides a comprehensive framework that is useful for the analysis and optimization of dc magnetic shields, serving as a theoretical and conceptual design guide as well as a starting point and benchmark for finite-element analysis. © 2014 Author(s). All article content, except where otherwise noted, is licensed under a Creative Commons Attribution 3.0 Unported License. [<http://dx.doi.org/10.1063/1.4873714>]

I. INTRODUCTION

Passive magnetic shielding systems typically use a concentric arrangement of thin shells of a high permeability material to divert magnetic field lines around a region of interest. The region within the shielding system consequently possesses a reduced local magnetic field.

While magnetic shielding is useful for a variety of applications, the most stringent requirements are found in high precision experiments where the limits of magnetometry technology are experienced or are themselves being studied. Some recent examples are in biomagnetism,^{1,2} electric dipole moment experiments,^{3,4} and atomic magnetometer development.⁵⁻⁷

Neutron electric dipole moment (EDM) experiments in particular suffer from a systematic effect relating to the accrual of geometric phase as neutrons and comagnetometer atoms sample the experimental volume.⁸⁻¹¹ This geometric phase effect is expected to present a dominant systematic effect in future neutron EDM experiments. To first approximation, the systematic correction is proportional to the first-order gradient along the direction of the applied magnetic field $\partial B_z/\partial z$. It is therefore important in these experiments both to limit and to characterize magnetic field gradients.

While the analysis and development of single- and multi-layer magnetic shields has been an important and active area of research for well over a century,¹²⁻²⁵ the focus in analytical treatments has been almost exclusively on shielding uniform magnetic fields. To the best of our knowledge, only Urankar and Oppelt²³ have explored the issue of passive magnetic shielding in gradient fields from an analytical perspective.

Sumner *et al.*²¹ provide an excellent overview of the history of magnetic shielding. Exact solutions for shields of concentric cylindrical and spherical shells of linear material^{12,13,15} have been simplified to approximate formulae valid in the limit of high magnetic permeability and thin

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shells^{14,16-21} as well as to provide axial shielding factors for cylindrical geometries. More recently, axial shielding in relation to shell spacing, end cap holes, and gaps between mating surfaces has been explored numerically.^{24,25} Analytic treatments of the quasi-static solutions have led to developments in external active compensation.²²

However, as mentioned above, these works considered only uniform applied fields. Urankar and Oppelt²³ analyzed the general multipole field (both as an external and internal source) for single spherical shields, and provided general shielding and reaction factors. They employed their results to analyze active magnetic compensation used in conjunction with magnetically shielded rooms. Quasi-static solutions valid in the dc limit were provided. We extend this work (in the dc limit) to multi-layer shields with spherical as well as infinite cylindrical geometry. For each, we consider the following situations:

1. *Externally applied fields.* Calculations are included for both single and multi-layer shields. The static shielding factor for general multipole fields is calculated internal to the innermost shell and is of principal interest. The field external to the shield is also calculated, and is useful for designing active magnetic compensation systems. This field is typically dominated by the response of the outermost layer (provided it is not near saturation) and the analysis is restricted to a single shell only.
2. *Internally applied fields.* In many cases, such as in EDM experiments, a highly homogeneous internal field is desired and this is generally supplied by a coil system internal to a set of magnetic shields. We consider here the impact of the innermost magnetic shield on internally-generated multipole fields, and calculate reaction factors by which the field internal to the coil system is amplified. This analysis provides further insight into the design of internal coil systems and complements existing work on current structures inside closed and open cylindrical shields with magnetic²⁶⁻³⁰ and superconducting boundary conditions.^{29,31-34}

We comment here on our primary new results:

- We report static shielding factors, interior reaction factors, and exterior response fields for single layer, infinitely-long cylindrical magnetic shields, exposed to general multipole transverse dc fields. This extends the work of Ref. 23 from spherical to cylindrical geometries. For the spherical case, we demonstrate agreement with Ref. 23. Our results for single-layer shields are useful for designing active magnetic compensation systems (in the case of exterior response fields) and internal coil systems (in the case of interior reaction factors), and we provide useful examples of this.
- We provide shielding factors for multi-layer shields in both cylindrical and spherical geometries for general multipole fields. One of our primary results is that higher-order multipole fields are always shielded better than the homogeneous field, a general result that should prove useful in applications requiring homogeneous fields. This extends previous work to general multipole fields, and extends the work of Ref. 23 to multi-layer shielding systems in the dc limit.
- Finally, we use a somewhat unique method of solution compared to previous authors, in that we consider the equivalent problem of bound surface currents. While the end results are of course equivalent, our approach may be useful in certain situations. We found, for example, that the consideration of surface currents gives a more direct conceptual link to the coil systems that one ultimately employs. This is demonstrated by examples of internal coils used to generate homogeneous fields.

Our work is valid for dc fields, general multipole sources (both internal and external to the magnetic shield), and any number of concentric shells (cylindrical or spherical). We provide an exact treatment valid for shells of any thickness and uniform, constant permeability μ . We also provide new approximate formulae in the high- μ , thin-shell limit, which we have now validated for all higher-order multipoles.

We proceed first by describing our method of solution. We then present general solutions for single and multiple concentric shields. We conclude with applications to some geometries of interest

in EDM experiments, which as noted above have very stringent requirements for magnetic field quality.

II. PROBLEM STATEMENT AND METHOD OF SOLUTION USING EQUIVALENT BOUND SURFACE CURRENTS

Two problems of particular geometry are solved here using standard cylindrical and spherical coordinates: (i) the interaction of the transverse, 2-dimensional magnetic field $\mathbf{B} = B_\rho(\rho, \phi)\hat{\rho} + B_\phi(\rho, \phi)\hat{\phi}$ with infinitely-long cylindrical shells, and (ii) the interaction of the general magnetic field $\mathbf{B} = B_\rho(\rho, \theta, \phi)\hat{\rho} + B_\theta(\rho, \theta, \phi)\hat{\theta} + B_\phi(\rho, \theta, \phi)\hat{\phi}$ with spherical shells.

As is commonly done to achieve analytic solutions for passive shielding problems, we restrict our analysis to shields of linear, homogeneous media, carrying no free current. Under such conditions, the response of a permeable object to an applied magnetic field can be recast in terms of bound current on the surfaces of the object. As a result, we take advantage of known formulae for the magnetic fields generated by cylindrical and spherical sheet currents³⁵⁻³⁷ to solve for the magnitudes of the unknown bound surface currents on sets of concentric magnetic shells. This is achieved by satisfying the boundary condition for the tangential component of the magnetic field at each shell surface. The continuity of the normal component of the magnetic field is already satisfied by the formulae from Refs. 35-37.

For a shield system comprising M concentric shells, there are $2M$ distinct surface currents contributing to the net magnetic field in each region. Satisfying the boundary condition

$$H_{\text{in}}^{\parallel} = H_{\text{out}}^{\parallel}$$

or

$$\frac{1}{\mu_{\text{in}}} B_{\text{in}}^{\parallel} = \frac{1}{\mu_{\text{out}}} B_{\text{out}}^{\parallel} \quad (1)$$

for the tangential component of the magnetic field results in a set of $2M$ simultaneous equations that determine the magnitudes of the unknown surface currents. By contrast, the typical means of solution using the magnetic scalar potential (e.g. Ref. 38) gives a set of $4M$ simultaneous equations, albeit resulting in a sparser matrix.

III. THE INFINITELY LONG CYLINDRICAL SHIELD

A. The 2D multipole field generated by a cylindrical current sheet

From Refs. 35-37, an axial surface current

$$\mathbf{K} = K \sin(n\phi) \hat{z} \quad (2)$$

with n -fold rotational symmetry ($n \geq 1$) bound to a cylindrical surface $\rho = a$ gives rise to the vector potential

$$\mathbf{A} = \mathcal{K} \frac{\sin(n\phi)}{n} \begin{cases} \rho^n \hat{z} & \rho < a \\ \frac{a^{2n}}{\rho^n} \hat{z} & \rho > a \end{cases}, \quad (3)$$

where $\mathcal{K} = \mu_0 K / (2a^{n-1})$ has units T/m^{n-1} . The introduction of \mathcal{K} , while not necessary, leads to a simplified notation for the determination of shielding factors, especially when multiple shields are considered. The magnetic field arising from Eq. (3) is

$$\mathbf{B} = \mathcal{K} \begin{cases} \rho^{n-1} [\cos(n\phi) \hat{\rho} - \sin(n\phi) \hat{\phi}] & \rho < a \\ \frac{a^{2n}}{\rho^{n+1}} [\cos(n\phi) \hat{\rho} + \sin(n\phi) \hat{\phi}] & \rho > a \end{cases}. \quad (4)$$

We use these results to solve the following problems.

B. A single cylindrical shield in an external field

Consider an infinitely-long cylindrical shield of inner radius R , thickness t , and permeability μ in the presence of an externally applied transverse magnetic field

$$\mathbf{B}_{\text{ext}} = G_n \rho^{n-1} [\cos(n\phi) \hat{\rho} - \sin(n\phi) \hat{\phi}] \quad (5)$$

with a magnitude gradient G_n in T/m^{n-1} . The case $n = 1$ corresponds to a uniform field, and $n > 1$ corresponds to higher-order multipole fields. By symmetry, the bound currents induced on the inner surface ($r_1 = R$) and outer surface ($r_2 = R + t$) of the magnetic shield have the same harmonic n as \mathbf{B}_{ext} and generate fields given by Eq. (4).

To find the coefficients \mathcal{K}_1 and \mathcal{K}_2 , representative of the bound surface current on the inner and outer surfaces of the shield, respectively, the boundary condition of Eq. (1) is applied to the azimuthal component B_ϕ of the net magnetic field. This results in the following system of equations:

$$(\mu + \mu_0) \mathcal{K}_1 + (\mu - \mu_0) \mathcal{K}_2 = -(\mu - \mu_0) G_n \quad (6)$$

$$(\mu - \mu_0) \left(\frac{r_1}{r_2} \right)^{2n} \mathcal{K}_1 + (\mu + \mu_0) \mathcal{K}_2 = (\mu - \mu_0) G_n, \quad (7)$$

which has solutions

$$\mathcal{K}_1 = -2G_n \frac{\mu(\mu - \mu_0)}{(\mu + \mu_0)^2 - (r_1/r_2)^{2n} (\mu - \mu_0)^2} \quad (8)$$

and

$$\mathcal{K}_2 = G_n \frac{\mu^2 - \mu_0^2 + (r_1/r_2)^{2n} (\mu - \mu_0)^2}{(\mu + \mu_0)^2 - (r_1/r_2)^{2n} (\mu - \mu_0)^2}. \quad (9)$$

Defining the shielding factor S as the applied field divided by the net internal field^{16,17,20,21} gives

$$\begin{aligned} S &= \frac{G_n}{\mathcal{K}_1 + \mathcal{K}_2 + G_n} \\ &= \frac{(\mu + \mu_0)^2 - (r_1/r_2)^{2n} (\mu - \mu_0)^2}{4\mu\mu_0} \end{aligned} \quad (10)$$

$$= 1 + \frac{(\mu - \mu_0)^2}{4\mu\mu_0} \left[1 - \left(\frac{r_1}{r_2} \right)^{2n} \right]. \quad (11)$$

In the limit $R \gg t$ and $\mu \gg \mu_0$, this reduces to

$$S \simeq 1 + \frac{\mu}{\mu_0} \frac{n}{2} \frac{t}{\bar{R}}, \quad (12)$$

where $\bar{R} = R + t/2$ is the average radius of the shield.

The results of Eqs. (11) and (12) for the $n = 1$ case (i.e., a uniform applied field) agree with previous work.^{12,16,17,20–22,39–41} The important new result here is the generalization to higher n , where we find that higher-order multipole fields are always shielded better than the $n = 1$ case. In the thin shield limit, in particular, the shielding factor increases proportional to n . Taking a linear combination of external fields and summing Eq. (5) over n , one would find that the interior shielded volume always becomes more uniform.

We now consider the *exterior* field $\mathbf{B}_{\text{shield}}$, defined as the additional field induced by the presence of the magnetic shield. An important consideration for active shielding systems (which feed back on measurements of the net magnetic field outside the passive shield assembly) is the perturbation $\mathbf{B}_{\text{shield}}$ superimposed on the applied field in the region $\rho > r_2$. From Eqs. (4), (8), and (9), the general

solution is

$$\mathbf{B}_{\text{shield}} = \frac{\mathcal{K}_1 r_1^{2n} + \mathcal{K}_2 r_2^{2n}}{\rho^{n+1}} [\cos(n\phi) \hat{\rho} + \sin(n\phi) \hat{\phi}], \quad (13)$$

which for $\mu \gg \mu_0$ reduces to

$$\mathbf{B}_{\text{shield}} = G_n \frac{r_2^{2n}}{\rho^{n+1}} [\cos(n\phi) \hat{\rho} + \sin(n\phi) \hat{\phi}]. \quad (14)$$

This in turn can be recast as

$$\mathbf{B}_{\text{shield}} = \frac{\mu_0 m'_n}{4\pi \rho^{n+1}} [\cos(n\phi) \hat{\rho} + \sin(n\phi) \hat{\phi}], \quad (15)$$

where $m'_n = 4\pi G_n r_2^{2n} / \mu_0$ is the $(n + 1)^{\text{th}}$ multipole moment per unit length defined by

$$\mathbf{A} = \frac{\mu_0 m'_n \sin(n\phi)}{4\pi n \rho^n} \hat{z} \quad (16)$$

for the vector potential outside a current-carrying cylinder from Eq. (3).

The result for the exterior field is important because it may also be applied to multi-layer shielding systems, since it is the response of the outermost shield that typically dominates. Furthermore, as in the magnetic shielding case above, the exterior field may be decomposed into multipoles. The results can therefore be used to decide the optimal placement of the magnetic sensors in an active compensation system. For example, the sensors can be placed selectively to accentuate sensitivity to particular multipoles, considering also the steeper suppression of higher-order multipoles with increasing ρ .

C. Multiple shields in an external field

Now consider a set of M concentric cylinders in an applied external field given by Eq. (5). The geometry is shown in Fig. 1, where our conventions for labelling are also described. The m -th cylinder has an inner radius R_m , an outer radius $R_m + t_m$, a thickness t_m and a permeability μ_m . There are now $2M$ bound surface currents that one must find. The i -th surface current \mathcal{K}_i resides on the *inner* surface of the m -th shield if i is odd (i.e., $i = 2m - 1$) and on its *outer* surface if i is even (i.e. $i = 2m$). The radial location of \mathcal{K}_i is thus defined as

$$r_i = \begin{cases} R_m & \text{for } i = \text{odd and } m = \frac{i+1}{2} \\ R_m + t_m & \text{for } i = \text{even and } m = \frac{i}{2}, \end{cases} \quad (17)$$

i.e. $r_1 = R_1$ is the inner surface of the innermost shield, $r_2 = R_1 + t_1$ is the outer surface of the innermost shield, $r_3 = R_2$ is the inner surface of the next-to-innermost shield, and so on.

Satisfying the boundary condition of Eq. (1) at each surface leads to the general system of equations

$$\mathbf{A}\mathcal{K} = G_n \mathbf{I}, \quad (18)$$

where $\mathbf{I} = [1, 1, \dots, 1]^T$, $\mathcal{K} = [\mathcal{K}_1, \mathcal{K}_2, \dots, \mathcal{K}_{2M}]^T$, and \mathbf{A} is a $2M \times 2M$ matrix with elements

$$a_{ij} = \begin{cases} (r_j/r_i)^{2n} & \text{for } j < i \\ U_m & \text{for } j = i = \text{odd and } m = \frac{i+1}{2} \\ V_m & \text{for } j = i = \text{even and } m = \frac{i}{2} \\ -1 & \text{for } j > i \end{cases}, \quad (19)$$

where

$$U_m = -V_m = -\frac{\mu_m + \mu_0}{\mu_m - \mu_0}, \quad (20)$$

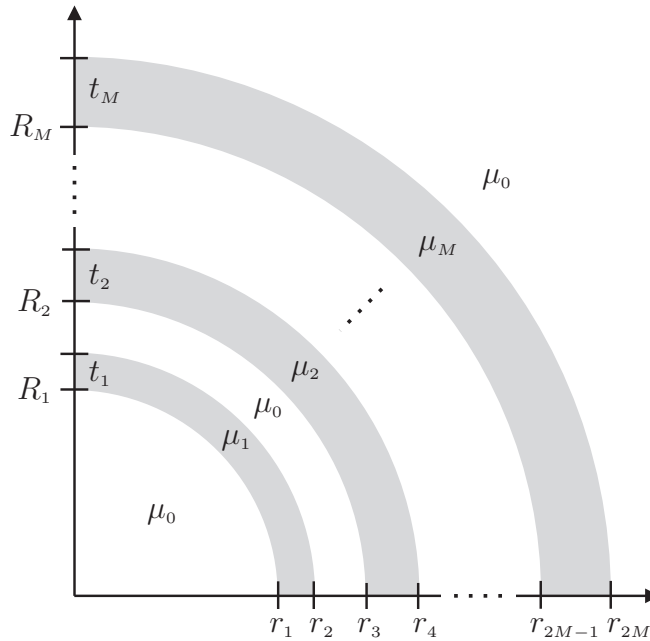


FIG. 1. Cross-sectional view (first quadrant) of M concentric cylindrical or spherical shields separated by free space. The material boundaries are located at radial positions r_1 through r_{2M} . The inner radius R , thickness t , and permeability μ of each shield is indicated on the drawing.

and r_i and r_j are defined per Eq. (17). The 2×2 diagonal submatrices of A correspond to Eqs. (6) and (7) for each individual, isolated shield. To illustrate, the explicit form of the general matrix A for $M = 2$ shields is

$$A = \begin{pmatrix} -\frac{\mu_1 + \mu_0}{\mu_1 - \mu_0} & -1 & -1 & -1 \\ \left(\frac{R_1}{R_1 + t_1}\right)^{2n} & \frac{\mu_1 + \mu_0}{\mu_1 - \mu_0} & -1 & -1 \\ \left(\frac{R_1}{R_2}\right)^{2n} & \left(\frac{R_1 + t_1}{R_2}\right)^{2n} & -\frac{\mu_2 + \mu_0}{\mu_2 - \mu_0} & -1 \\ \left(\frac{R_1}{R_2 + t_2}\right)^{2n} & \left(\frac{R_1 + t_1}{R_2 + t_2}\right)^{2n} & \left(\frac{R_2}{R_2 + t_2}\right)^{2n} & \frac{\mu_2 + \mu_0}{\mu_2 - \mu_0} \end{pmatrix}.$$

Returning now to the general case, the total combined shielding factor for M concentric cylindrical shields is

$$S_{\text{tot}} = G_n \left(G_n + \sum_{i=1}^{2M} \mathcal{K}_i \right)^{-1}, \tag{21}$$

with the \mathcal{K}_i determined from Eqs. (18) and (19). An algebraic scheme for the solution of Eq. (21) is given by Wills,¹² and one can show that our result agrees with his explicit formulation of S_{tot} for double and triple cylindrical shields of the same permeability μ . With the generating formulae of Eq. (19), Eq. (18) is also readily coded and solved using any number of computer programs designed for symbolic or numeric computation.

Numerical solutions of Eq. (21) for multi-layer shields, along with approximate formulae valid for the small- t , high- μ limit, will be discussed in Sec. V. A key generic feature will be the greater shielding of higher-order multipole fields.

D. A single cylindrical shield with an internal coil

We now turn to the study of coil systems internal to the magnetic shield system.^{26–30} In this case, the perturbation of the internal field is dominated by the innermost magnetic shield. We therefore consider here a single cylindrical shield in order to simplify the discussion.

Consider an applied surface current \mathbf{K} of the form Eq. (2) on a coaxial cylindrical surface $\rho = a$ inside a single shield of inner radius $r_1 = R$, outer radius $r_2 = R + t$, and permeability μ . Solving boundary conditions gives the following system of equations:

$$(\mu - \mu_0) \left(\frac{a}{r_1}\right)^{2n} \mathcal{K}_a - (\mu + \mu_0) \mathcal{K}_1 - (\mu - \mu_0) \mathcal{K}_2 = 0 \quad (22)$$

$$(\mu - \mu_0) \left(\frac{a}{r_2}\right)^{2n} \mathcal{K}_a + (\mu - \mu_0) \left(\frac{r_1}{r_2}\right)^{2n} \mathcal{K}_1 + (\mu + \mu_0) \mathcal{K}_2 = 0, \quad (23)$$

where $\mathcal{K}_a = \mu_0 K / (2a^{n-1})$. The equations are again solved for \mathcal{K}_1 and \mathcal{K}_2 .

The ratio of the field in the region $\rho < a$ with and without the shield present may then be calculated. We call this ratio the reaction factor C , in keeping with the terminology of Ref. 23. The result is

$$C = \frac{\mathcal{K}_a + \mathcal{K}_1 + \mathcal{K}_2}{\mathcal{K}_a} \quad (24)$$

$$= 1 + \left(\frac{a}{r_1}\right)^{2n} \frac{(\mu - \mu_0)(\mu + \mu_0) \gamma_n}{4\mu\mu_0 + (\mu - \mu_0)^2 \gamma_n} \quad (25)$$

where $\gamma_n \equiv 1 - (r_1/r_2)^{2n}$. In the limit $\mu \gg \mu_0$ this reduces to

$$C = 1 + \left(\frac{a}{R}\right)^{2n}, \quad (26)$$

and one sees that the internal field is augmented more strongly for small n than it is for large n since $a < R$. In the limit $a = R$, the reaction factor is identically 2, independent of n .

These results are applied to a sample internal coil design in Sec. V. A key feature here will be that internally-generated fields are in general more homogeneous with the shield than without, but that optimal homogeneity can be achieved for a particular geometrical factor a/R .

IV. THE SPHERICAL SHIELD

A. The zonal multipole field generated by a spherical current sheet

In general, any surface current bound to a sphere, and its resulting magnetic field, can be written in terms of spherical harmonics of order m and degree n .^{35,36} One can show, however, that the resulting equations arising from the boundary conditions on the tangential components of the magnetic field (i.e., B_θ and B_ϕ) are independent of the order m of the spherical harmonic. Without loss of generality, then, we can restrict the analysis of spherical shields to zonal surface currents and fields only (i.e., ϕ -independent, $m = 0$), a simplification also noted by Urankar and Oppelt.²³ This also means that the following results can be applied to cases where tesseral components ($m > 0$) do exist in the fields and currents, which is extremely valuable from the point of view of coil design, where the general spherical harmonics can be used as *building blocks* to produce a desired magnetic field.⁴²

From Refs. 35 and 36, the zonal surface current

$$\mathbf{K} = K P_n^1(u) \hat{\phi} \quad (27)$$

bound to a spherical surface $r = a$ gives rise to the vector potential

$$\mathbf{A} = \mathcal{K} P_n^1(u) \begin{cases} r^n \hat{\phi} & r < a \\ \frac{a^{2n+1}}{r^{n+1}} \hat{\phi} & r > a \end{cases}, \quad (28)$$

where $P_n^1(u)$ is the associated Legendre function of order 1 and degree n , $u = \cos \theta$, and the coefficient $\mathcal{K} = \mu_0 K / ((2n + 1)a^{n-1})$ has units T/m^{n-1} . The magnetic field arising from Eq. (28) is

$$\mathbf{B} = \mathcal{K} \begin{cases} r^{n-1} (n+1) [nP_n(u) \hat{\mathbf{r}} - P_n^1(u) \hat{\boldsymbol{\theta}}] & r < a \\ \frac{a^{2n+1}}{r^{n+2}} n [(n+1)P_n(u) \hat{\mathbf{r}} + P_n^1(u) \hat{\boldsymbol{\theta}}] & r > a \end{cases}, \quad (29)$$

where $P_n(u)$ is the Legendre function of degree n . We use these results to solve the following problems.

B. A single spherical shield in an external field

Consider a spherical shield of inner radius $r_1 = R$, outer radius $r_2 = R + t$, and permeability μ in the presence of an externally applied magnetic field

$$\mathbf{B}_{\text{ext}} = G_n r^{n-1} [n(n+1)P_n(u) \hat{\mathbf{r}} - (n+1)P_n^1(u) \hat{\boldsymbol{\theta}}] \quad (30)$$

with a magnitude gradient G_n in T/m^{n-1} . The method of analysis follows exactly as above, and the solution of the boundary conditions on the tangential field B_θ leads to the general shielding factor

$$S = 1 + \frac{(\mu - \mu_0)^2}{\mu \mu_0} \frac{n(n+1)}{(2n+1)^2} \left[1 - \left(\frac{r_1}{r_2} \right)^{2n+1} \right], \quad (31)$$

which agrees with Ref. 23. In the limit of a thin shield ($t \ll R$) with large permeability ($\mu \gg \mu_0$), the shielding factor can be approximated as

$$S \simeq 1 + \frac{\mu}{\mu_0} \frac{n(n+1)}{2n+1} \frac{t}{R}. \quad (32)$$

The results of Eqs. (31) and (32) for the $n = 1$ case (i.e., a uniform applied field) agree with previous authors.^{12,15,38} Similar to the cylindrical case, higher-order multipole fields are shielded progressively better, and in the large- n limit the shielding factor again becomes proportional to n . In cases where the applied magnetic field would be a linear combination of fields with differing n , the magnetic field internal to the shield would therefore always be more uniform than the applied field.

We now again consider the exterior field induced by the presence of the magnetic shield in the region $r > r_2$. In this case, the perturbation of the external field by a spherical shield of $\mu \gg \mu_0$ is

$$\begin{aligned} \mathbf{B}_{\text{shield}} &= G_n (n+1) \frac{r_2^{2n+1}}{r^{n+2}} [(n+1)P_n(u) \hat{\mathbf{r}} + P_n^1(u) \hat{\boldsymbol{\theta}}] \\ &= \frac{\mu_0 m_n}{4\pi r^{n+2}} [n(n+1)P_n(u) \hat{\mathbf{r}} + nP_n^1(u) \hat{\boldsymbol{\theta}}], \end{aligned} \quad (33)$$

where $m_n = 4\pi G_n r_2^{2n+1} (n+1) / (n\mu_0)$ is the $(n+1)^{\text{th}}$ multipole moment defined by

$$\mathbf{A} = \frac{\mu_0 m_n}{4\pi r^{n+1}} P_n^1(u) \hat{\boldsymbol{\phi}} \quad (34)$$

for the vector potential outside a current-carrying sphere from Eq. (28).

C. Multiple spherical shields in an external field

For M concentric shields, we again have the same system of equations $\mathbf{A}\mathcal{K} = G_n \mathbf{I}$ where now the general matrix elements of \mathbf{A} are

$$a_{ij} = \begin{cases} \frac{n}{n+1} (r_j/r_i)^{2n+1} & \text{for } j < i \\ U_m & \text{for } j = i = \text{odd and } m = \frac{i+1}{2} \\ V_m & \text{for } j = i = \text{even and } m = \frac{i}{2} \\ -1 & \text{for } j > i \end{cases}, \quad (35)$$

with

$$U_m = -\frac{(n+1)\mu_m + n\mu_0}{(n+1)(\mu_m - \mu_0)}, \quad (36)$$

$$V_m = \frac{n\mu_m + (n+1)\mu_0}{(n+1)(\mu_m - \mu_0)}, \quad (37)$$

and r_i and r_j are defined per Eq. (17). In general the total combined shielding factor for M concentric spherical shields is given by Eq. (21) with the \mathcal{K}_i determined from Eqs. (18) and (35).

One can show that the general shielding factor of Eq. (21) reduces to the explicit formula for double and triple spherical shields of the same permeability.^{12,15} We calculate sample results for multi-layer magnetic shields in Sec. V. Similar to the cylindrical case, a generic feature will be the greater shielding of higher-order multipole fields.

D. A single spherical shield with an internal coil

Again driven by the desire to create an internal coil system that generates a homogeneous field, we consider internal coils wound on a spherical surface inside the magnetic shielding system. As in the cylindrical case, the modification of the internal field will be dominated by the response of the innermost magnetic shield, and we restrict the analysis to a single spherical shield.

Consider an applied surface current \mathbf{K} of the form Eq. (27) on $r = a$ inside a spherical shield of inner radius $r_1 = R$, outer radius $r_2 = R + t$, and permeability μ . Following the method laid out in Sec. III D, the reaction factor giving the ratio of field in the region $r < a$ with and without the shield is

$$C = 1 + \left(\frac{a}{r_1}\right)^{2n+1} \times \frac{n(\mu - \mu_0)(n(\mu + \mu_0) + \mu_0)\gamma_n}{(2n+1)^2\mu\mu_0 + n(n+1)(\mu - \mu_0)^2\gamma_n}, \quad (38)$$

where now $\gamma_n = 1 - (r_1/r_2)^{2n+1}$. In the limit $\mu \gg \mu_0$ this reduces to

$$C = 1 + \frac{n}{n+1} \left(\frac{a}{R}\right)^{2n+1}. \quad (39)$$

These results agree with Ref. 23. An interesting difference with the cylindrical case is the prefactor $n/(n+1)$ preceding the second term. Because of it, there is a cross-over behaviour in the relative magnitudes of the reaction factors and one finds that higher-order fields become augmented more strongly (not less) by the presence of the shield as $a/R \rightarrow 1$. This is discussed further in Sec. V C.

V. RESULTS AND APPLICATIONS

A. Multiple shields: Numerical results and useful approximations

Most practical interests lie in the construction of multiple shields made of thin material ($t_m \ll R_m$) with large permeability ($\mu_m \gg \mu_0$). Many previous authors provided approximations for designing shields in this regime. A well-known result, for the total shielding factor S_{tot} for well-separated shields,^{16,18,21} is generalized to any n as follows:

$$S_{\text{tot}} \simeq \prod_{m=1}^{M-1} S_M S_m \left[1 - \left(\frac{\bar{R}_m}{\bar{R}_{m+1}}\right)^\beta \right], \quad (40)$$

where S_m is the shielding factor of the m -th shield (from Eq. (12) or (32)), \bar{R}_m is the average radius of the m -th shield, and the exponent β equals $2n$ for cylinders and $2n+1$ for spheres.

In Figs. 2 and 3 we compare Eq. (40) with the general result of Eq. (21) for cylindrical and spherical shields, respectively. We analyze a shield geometry that is likely typical of many applications: four concentric shields each of the same thickness $t = \frac{1}{16}$ inches ~ 1.6 mm (a standard

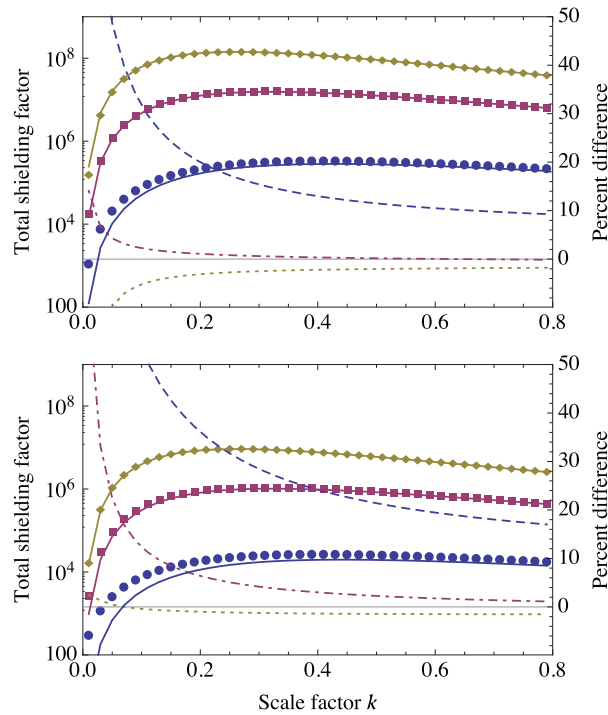


FIG. 2. The total shielding factor of four concentric cylindrical shells of permeability $\mu/\mu_0 = 4 \times 10^4$ (top) and 2×10^4 (bottom) determined from Eq. (21) for applied fields with $n = 1$ (blue circles), 2 (red squares), and 3 (yellow diamonds). The solid lines are the results of Eq. (40). The right ordinate axis gives the percent difference between Eqs. (21) and (40) for $n = 1$ (dashed line), 2 (dot-dashed line), and 3 (dotted line).

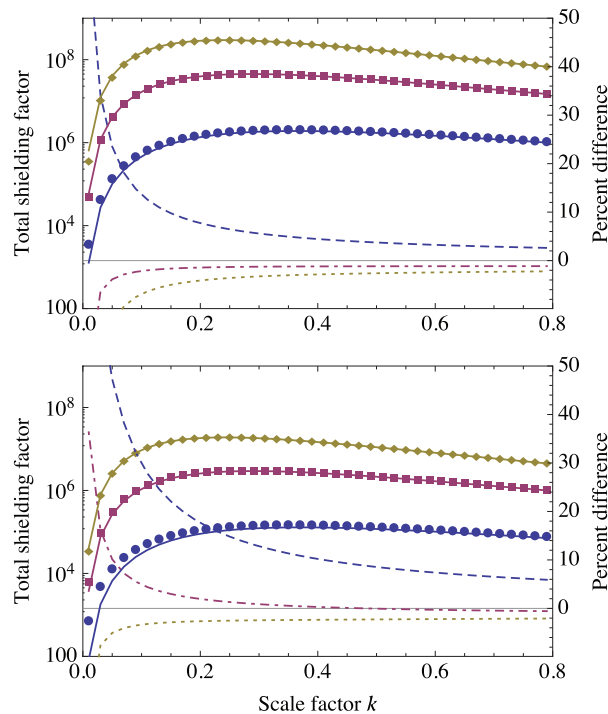


FIG. 3. The total shielding factor of four concentric spherical shells of permeability $\mu/\mu_0 = 4 \times 10^4$ (top) and 2×10^4 (bottom) determined from Eq. (21) for applied fields with $n = 1$ (blue circles), 2 (red squares), and 3 (yellow diamonds). The solid lines are the results of Eq. (40). The right ordinate axis gives the percent difference between Eqs. (21) and (40) for $n = 1$ (dashed line), 2 (dot-dashed line), and 3 (dotted line).

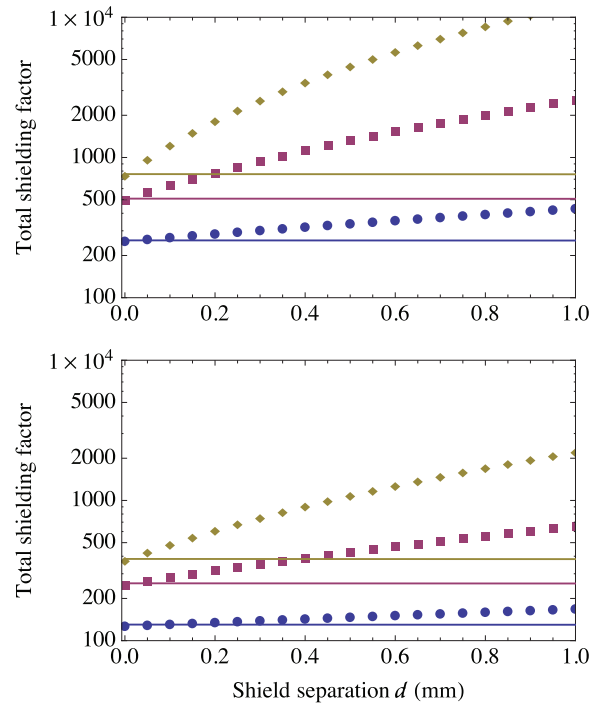


FIG. 4. The total shielding factor of four closely-spaced concentric cylindrical shells of permeability $\mu/\mu_0 = 4 \times 10^4$ (top) and 2×10^4 (bottom) determined from Eq. (21) for an applied field of $n = 1$ (blue circles), 2 (red squares), and 3 (yellow diamonds). The solid lines are the results of Eqs. (41) and (12), and are a very weak inverse function of d .

size) with a radius $R_1 = 0.5$ m for the inner most shield. All shields have the same permeability and we examine two specific cases: $\mu = 2 \times 10^4 \mu_0$ and $\mu = 4 \times 10^4 \mu_0$. The shield spacings are set by a single geometrical scale factor k , such that the inner radius of the m -th shield is $R_m = (1 + k)^{m-1} R_1$.

A key feature is that higher-order multipole fields are always progressively suppressed as n increases. For example, for the four-layer shield explored here, the shielding factor for $n = 2$ is of order 10^2 greater than for $n = 1$. The optimal choice of scale factor k is relatively independent n . Furthermore, the approximate formulae of Eq. (40) appear to be even more accurate for higher n than for the $n = 1$ case. This is shown in Figs. 2 and 3 as a percent difference from the exact result.

For closely packed cylindrical and spherical shields, on the other hand, a useful approximation for the total shielding factor is

$$S_{\text{tot}} \simeq \sum_{m=1}^M S_m, \quad (41)$$

which is now validated for all n . For shields that just touch, Eq. (41) correctly approximates the shielding factor of a single shield with thickness equivalent to the total thickness of the shielding material. As an example, we show in Fig. 4 plots of S_{tot} as a function of a small separation d between each of the four concentric cylindrical shields discussed above. Similar results hold for spherical shields.

At $d = 0$, we find that for the range of parameters studied here, Eq. (41) over predicts S_{tot} by $\sim 3 - 5\%$ compared to the exact result of Eq. (21). This is reduced slightly if one uses Eq. (11) instead of Eq. (12) for S_m . We also point out, that as expected, the value of S_{tot} from Eq. (21) for the four shields with $d = 0$ agrees exactly with Eq. (11) for a single shield that is four times as thick.

The results of Fig. 4 also highlight the importance of sufficiently separating the shields. An interesting observation is that the shielding of higher-order fields increases dramatically with n even for sub-millimetre shield spacing. This may argue for subdividing shields further, possibly with thin

interstitial nonmagnetic layers, if desiring particularly to reduce gradients with relatively less impact on the uniform field case. For example, an application requiring better control of $n > 1$ could use four well-separated shields to reduce $n = 1$, but each of those four shields could comprise thinner layers separated by plastic sheet, say, to augment further the shielding of higher-order fields. It should be noted, of course, that in practice the fabrication, handling and magnetic saturation of very thin mu-metal shells could make such a scheme particularly challenging.

B. The external physical dipole

The source of external gradient fields can often be linked to some nearby *dipole* – a research magnet, a steel door, or even a passing vehicle.³ A very important example to study then is the field of the physical dipole, or current loop, expressed in spherical coordinates. More complicated magnetic structures can often be modelled from a superposition of such loops or, as mentioned before, using a decomposition into general spherical harmonics.⁴²

Here we consider a circular loop of radius r_c carrying current I that is co-axial with the z -axis and lying in the plane $z = z_c$. The loop can also be viewed as lying on a sphere of radius $a = \sqrt{r_c^2 + z_c^2}$ at the polar angle $\alpha = \tan^{-1} r_c/z_c$. The magnetic field of the loop can be decomposed into zonal harmonics^{35,36} and therefore its interaction with spherical shields is easily determined using the results of Sec. IV.

For example, in the region $r < a$ the magnetic field components of the loop are

$$\begin{pmatrix} B_r \\ B_\theta \end{pmatrix} = \frac{\mu_0 I \sin \alpha}{2a} \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^{n-1} P_n^1(\cos \alpha) \times \begin{pmatrix} -P_n(\cos \theta) \\ P_n^1(\cos \theta) \end{pmatrix}. \quad (42)$$

All that remains is to multiply each multipole component of this field by the appropriate shielding factor from Sec. IV C to determine the net interior field. Furthermore, the reflected exterior field, dominated by the response of the outermost magnetic shield, may be determined by applying the results of Sec. IV B to each multipole component. An appropriate sensor and coil system to effectively cancel particularly problematic external dipoles of this sort can then be devised.

C. Generation of a uniform internal field

A critical requirement of many experiments⁸⁻¹¹ is the generation of a highly uniform magnetic field in the inner volume of a passive shield system. If the coils used to generate this field are not *self-shielded* in some manner³³ they will couple strongly to the shields. This coupling, if taken into account properly, can be used advantageously to improve the field homogeneity over the case where no passive shielding is present.²⁶⁻³⁰ In order to accentuate this point, and to illustrate the usefulness of our formulation, we present here two simple, canonical examples – the saddle-shaped coil and the Helmholtz coil.

In the cylindrical case, a saddle-shaped coil can be used to produce a transverse field with a dominant $n = 1$ term near $\rho = 0$. For a very long (infinite) coil, the $n = 3$ term can be eliminated by placing the four axial current paths at $\phi = \pm \frac{\pi}{3}$ and $\pm \frac{2\pi}{3}$.³³ An end view of the geometric arrangement of the currents is shown in Fig. 5. Using the results of Ref. 33 along with Eq. (26) from above, the components of the internal field of such a coil inside a high- μ cylindrical shield are

$$\begin{pmatrix} B_\rho \\ B_\phi \end{pmatrix} = \frac{2\mu_0 I}{\pi a} \sum_{n=1,5,7,\dots}^{\infty} \sin\left(n \frac{\pi}{3}\right) \left(\frac{\rho}{a}\right)^{n-1} \times \left[1 + \left(\frac{a}{R}\right)^{2n} \right] \begin{pmatrix} \cos n\phi \\ -\sin n\phi \end{pmatrix}, \quad (43)$$

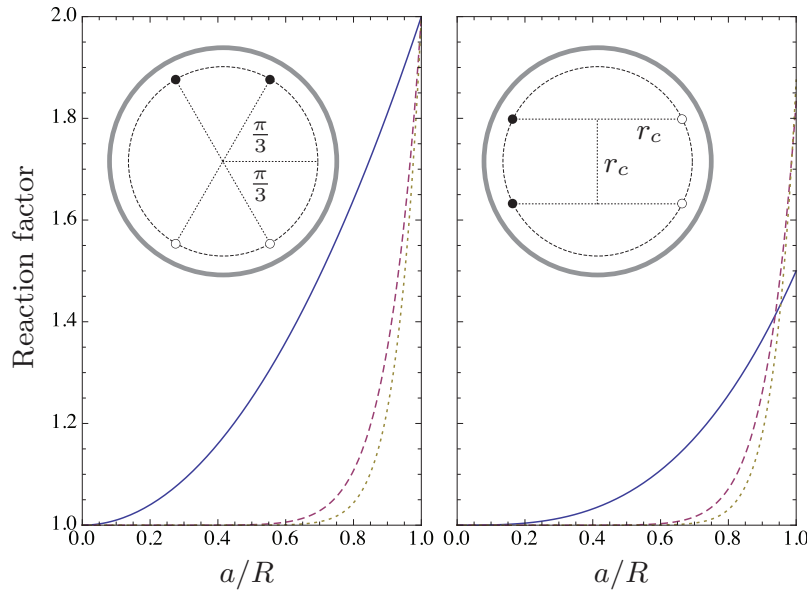


FIG. 5. The reaction factor for $n = 1$ (solid line), 5 (dashed line), and 7 (dotted line) from Eqs. (26) and (39) for the cylindrical (left) and spherical (right) case, respectively. Insets: Schematic of a saddle coil (left) and Helmholtz coil (right) located on the radius a (dashed line) inside a shield (thick gray line) of inner radius R . The dotted lines define the geometry of the coil and give the locations of the current (circles). The closed (opened) symbols indicate current flow out of (into) the page.

where the sum is over odd n not equal to an integer multiple of 3. Provided that the coil is not located directly on the inner surface of the shield (i.e., $a < R$) the resulting field is always more homogeneous than an unshielded coil ($R \rightarrow \infty$), because the term in square braces – the reaction factor – is greatest for $n = 1$ and decreases for all higher-order terms. It is informative to contrast this with the case of a superconducting or perfectly conducting shield, where free currents induced on the inner surface of the shield act to reduce the field in the region $\rho < a$. In this case, the term in square braces is replaced by $[1 - (a/R)^{2n}]$ (as given in Eq. (31) of Ref. 33) and the field becomes less homogeneous in the presence of the shield. This is also discussed in Refs. 29 and 32.

As can be seen from the plot in Fig. 5, however, there must exist a value of a/R for which the ratio of the reaction factor for $n = 5$ compared to that for $n = 1$ is a minimum. One can show that this occurs at $a/R = 0.7784$ and that for a coil located at this position the $n = 5$ term is $\sim 33\%$ lower relative to the $n = 1$ term compared to the unshielded case. This result demonstrates that field homogeneity can be obtained not only by appropriate coil design but also by a judicious choice of a for the location of the coil inside the shield.

Turning to the spherical case, a Helmholtz coil can be used to produce an axial field with a dominant $n = 1$ term. The coil is constructed from two current loops located at $z = \pm r_c/2$ (as shown in Fig. 5), or equivalently at polar angles α and $\pi - \alpha$ where $\sin \alpha = 2/\sqrt{5}$ and $\cos \alpha = 1/\sqrt{5}$. Since $\sin(\pi - \alpha) = \sin \alpha$, $\cos(\pi - \alpha) = -\cos \alpha$, and $P_n^1(u)$ is an even (odd) function of u for odd (even) degree n , only the odd n terms of Eq. (42) contribute to net field. Furthermore, since $P_3^1(\pm 1/\sqrt{5})$ is uniquely zero – thereby eliminating the $n = 3$ term – the field components can be written as

$$\begin{pmatrix} B_r \\ B_\theta \end{pmatrix} = \frac{2\mu_0 I}{\sqrt{5} a} \sum_{n=1,5,7,\dots}^{\infty} \left(\frac{r}{a}\right)^{n-1} P_n^1(1/\sqrt{5}) \times \begin{pmatrix} -P_n(\cos \theta) \\ P_n^1(\cos \theta) \end{pmatrix}. \quad (44)$$

The expansion of Eq. (44) in $r = z$ at $\theta = 0$ gives the following leading order terms – corresponding here to $n = 1$ and 5 – for the field along the central axis:

$$B_z = B_c \left(1 - \frac{144}{125} \left(\frac{z}{r_c} \right)^4 + \dots \right), \quad (45)$$

where $B_c = \mu_0 I / r_c \times (4/5)^{3/2}$ is the well-known central field of a Helmholtz coil.

If the coil is now placed inside a high- μ spherical shield of inner radius $R > a$, the relative strength of these terms will vary according to Eq. (39), as shown in Fig. 5. One can show that for $a/R = 0.7817$ the ratio of the reaction factor for $n = 5$ compared to that for $n = 1$ is a minimum and the relative strength of the z^4 term is reduced by $\sim 15\%$ compared to the unshielded coil. For $a/R > 0.9381$, where the reaction factor of the $n = 5$ term becomes greater than that for $n = 1$, the homogeneity near the origin is in fact degraded. This highlights the care that must be taken in designing shield-coupled coils, even when considering ideal geometries.

VI. CONCLUSION

In this paper, we have provided solutions for the interaction of static gradient fields with passive magnetic shields comprising concentric spherical or infinitely-long cylindrical shells of linear material. The results are general to any such shields in the presence of any dc magnetic field distribution that can be decomposed into the appropriate multipoles. For externally-generated fields, higher-order multipole components are always shielded progressively better than the uniform field case. Such a simple trend does not exist for the reaction factors of internally-generated fields, however, and one finds that higher-order multipole components can in fact be augmented more strongly than the uniform field. This further highlights the importance of incorporating the shield response into the design of internal coil structures.^{26–34}

We have also provided here a few examples that demonstrate the utility of this work, using our formulae to analyze coil systems located both inside and outside a magnetic shield. In future work, we intend to study the passive shielding of gradient fields when more realistic shield geometries and magnetic properties are considered. Such problems do not generally afford analytic solutions, and one naturally resorts to finite element analysis (FEA) codes to conduct such studies. As a result, we envision first benchmarking FEA code to the analytic formulae provided here for idealized models.

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