A Characterization of Fractionally Well-Covered Graphs

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Abstract. A graph is called well-covered if every maximal independent set has the same size. One generalization of independent sets in graphs is that of a fractional cover - attach nonnegative weights to the vertices and require that for every vertex the sum of all the weights in its closed neighbourhood be at least 1. In this paper we consider and characterize fractionally well-covered graphs.

Plummer [5] called a graph *well-covered* if every maximal independent set is of the same size. If a graph is well-covered then any version of the greedy algorithm will find a maximum sized independent set. It appears that characterizing well-covered graphs is difficult (see [1] and [3] for example) and few results are known. Staples [7] characterized those graphs in which every maximal independent set contains exactly half of the vertices. (See also Favaron [2] and Ravindra [6].) Campbell and Plummer [1] characterized cubic, 3-connected, planar, well-covered graphs and Finbow, Hartnell and Nowakowski [3] characterized well-covered graphs of girth 5 or greater. A graph is called *well-dominated* if every minimal dominating set has the same size. Since a maximal independent set is also dominating, well-dominated graphs form a subset of well-covered graphs. (See [4] for more examples.)

Let $G = (V, E)$ be a graph. The closed neighbourhood of a vertex x is $N[x] = \{y|y \text{ is adjacent to } x\} \cup \{x\}$. Call an assignment $p : V \rightarrow \mathbf{R}^{\geq 0}$ of the vertices of G to the nonnegative real numbers a *weighting*; denote the sum of the weights assigned to the vertices of a set S by $W(S,p)$ - in particular, $W(G,p)$ will be called the *graph weight* and $W(N[x],p)$ the *neighbourhood weight*; a weighting p will be called a fractional cover if $W(N[v],p) \geq 1$ holds for every vertex v of G; call a fractional cover *minimal* if no weight can be reduced and the weighting remain a fractional cover. Both maximal independent sets and minimal dominating sets can be interpreted as minimal fractional covers. This is done by assigning to every vertex in the set the weight 1, all others the weight 0 and the graph weight in each case is the cardinality of the set. A graph G is *fractionally well-covered* if the graph weight is the same for every minimal fractional cover. (It is easy to see that fractionally well- covered graphs are also well-dominated.)

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In Figure 1a, the given fractional cover has a graph-weight of 5/3 but giving a weight of 1 to each vertex in a maximum independent set would give a minimal fractional cover with graph weight 2. In any fractional cover of the graph in Figure 1b, however, the neighbourhood weight of both x and y is at least 1 and since any minimal fractional cover would have graph weight at most 2 then the graph is fractionally well-covered. A vertex will be called simplicial if $N[x]$ is a complete subgraph; a graph G will be called *simplicial* if V can be partitioned into vertex disjoint, maximal, complete subgraphs where each subgraph contains at least one simplicial vertex. We now have

Theorem. *A graph is fractionally well-covered if and only if it is simplicial.*

Proof: Let G be a simplicial graph and $H_1, H_2, \ldots, H_k$ be the decomposition of G into disjoint complete subgraphs. Let $x_i \in H_i, i = 1, 2 \ldots, k$ be simplicial vertices. Let $p : V \longrightarrow \mathbf{R}^{\geq 0}$ be a minimal, fractional cover of G. Suppose for some i, $1 \leq i \leq k$, and for some $m > 0$, that $W(H_i, p) = 1 + m$. Choose $y \in H_i$ with $p(y) > 0$. Define the following weighting $p'(x) = p(x)$, if $x \neq y$ and $p'(y) = p(y) - min\{p(y), m\}$. This new weighting is a fractional cover since for any $j \neq i$, $1 \leq j \leq k$, and for any $x \in H_i$, then $W(N[x], p') \geq W(N[x_j], p')$ $= W(H_j, p) \geq 1$ and for any $x \in H_i$, $W(N[x], p') \geq W(N[x_i], p') \geq 1$. It follows therefore that in a minimal, fractional cover, each H_i contributes exactly 1 to the graph weight, i.e. all minimal, fractional covers of G have graph weight equal to k.

In order to prove the converse, we first prove a preliminary result.

Claim. *For $i = 1, 2, \ldots, k$, let $p_i : V \longrightarrow \mathbf{R}^{\geq 0}$, be fractional covers of G. Then $p : V \longrightarrow \mathbf{R}^{\geq 0}$ defined by $p(v) = (p_1(v) + p_2(v) + \cdots + p_k(v))/k$ is also a fractional cover.*

Proof of Claim: Let $x \in V$ then $W(N[x], p) = (W(N[x], p_1) + W(N[x], p_2) + \cdots + W(N[x], p_k))/k$. For each i, $i = 1, 2, \ldots, k$, p_i is fractional cover and therefore $W(N[x], p_i) \geq 1$. It follows then that $W(N[x], p) \geq 1$ and so p is a fractional cover.

We finish the proof of the theorem by induction. Let G be a smallest fractionally well-covered graph that is not a simplicial graph.

First, suppose that G contains no simplicial vertices. This means that every vertex is adjacent to two mutually non-adjacent vertices. For every vertex $x \in G$, take a maximal independent set $J(x)$ which includes at least two vertices adjacent to x; let p_x be the associated fractional cover. Let $p : V \longrightarrow \mathbf{R}^{\geq 0}$ be the weighting defined by $p(v) = [\sum_{x \in V} p_x(v)]/|V|$. By the Claim, this is a fractional cover and, in addition, we also have that for all $x \in V$, $W(G, p) = W(G, p_x)$. For distinct vertices v and x, note that $W(N[x], p_v) \geq 1$ but also that $W(N[x], p_x) \geq 2$. That is, for each vertex x there is a maximal independent set which has two vertices adjacent to x and so summing over the maximal independent sets, we see that x is adjacent or equal to at least $|V| + 1$ many members of independent sets. It follows then that $W(N[x], p) \geq (|V| + 1)/|V|$. Consequently this fractional cover is not minimal since any one vertex could have its weight reduced by $1/|V|$ and the new weighting would still be a fractional cover.

Suppose now that in G, there is a maximal complete subgraph H_0 with a simplicial vertex x. Therefore $G - H_0$ is also fractionally well-covered. By induction on $|V|$, $G - H_0$ is simplicial. Let $H_1, H_2, \ldots, H_k$ be a partition of $G - H_0$ into disjoint maximal, complete subgraphs of $G - H_0$, where each H_i contains a simplicial vertex x_i. Note that G is now partitioned into complete subgraphs. For $i = 1, 2, \ldots, k$, let F_i be a maximal, complete subgraph of G that contains H_i. Note that for each i, $F_i - H_i \subset H_0$. Now $F_1, F_2, \ldots, F_k$ and H_0 do not form an appropriate partition of G since this would contradict the assumption about G. Therefore, for some i, either $F_i - H_i$ is nonempty or F_i has no simplicial vertex.

In first case, since F_i is a proper superset of H_i then there exists $z \in F_i \cap H_0$. Let J be a maximal independent set of $G - F_i \cap H_0$. Now, $J \cup \{z\}$ is a dominating set of G and $J \cup \{x, x_i\}$ is a maximal independent set of G. This implies that G is not well-dominated and hence contradicts the assumption that G is fractionally well-covered.

Therefore, it follows that for some i, say $i = k$, F_k has no simplicial vertices in G. Now, every vertex in F_k must be adjacent to some vertex not in F_k. Note that if $y \in F_k$ and z is adjacent to y but z is not in F_k then z is in one of the complete subgraphs $H_1, \ldots, H_{k-1}, H_0$. For every vertex y in F_k, form a set by taking a vertex z (say in $H_i, 0 \leq i \leq k - 1$) adjacent to y but not in H_k, together with a vertex in H_k not adjacent to z and one vertex from each of $H_0, H_1, \ldots, H_{k-1}$ except for H_i. This set is a minimal dominating set since it dominates every vertex in G and has $k + 1$ vertices. Call the weighting associated with this dominating set p_y (p_y is a minimal fractional cover). Consider the weighting $p : V \longrightarrow \mathbf{R}^{\geq 0}$ defined by $p(v) = [\sum_{y \in H_k} p_y(v)]|H_k|$. We claim that this new weighting is not minimal. For each $y \in H_k$, $W(N[y], p) \geq (|H_k| + 1)/|H_k|$ since in each weighting p_z, we have $W(N[y], p_z) \geq 1$ but $W(N[y], p_y) \geq 2$. Now choose a vertex in H_k with positive p-weight and reduce its weight by $1/|H_k|$. The new neighbourhood weights for vertices in H_k are still at least 1. The neighbourhood weights for the other vertices are also still at least 1 because in each weighting p_y, each vertex $v \in H_i, i < k$ is either in the corresponding dominating set or else is adjacent to a vertex in H_i. This new weighting has a smaller graph weight than the original weightings hence none of the original weightings were minimal.

It follows then that the postulated graph G does not exist and the result is proved.

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