DEJEAN’S CONJECTURE HOLDS FOR N ≥ 27∗, ∗∗

JAMES CURRIE1 AND NARAD RAMPERSAD1

Abstract. We show that Dejean’s conjecture holds for n ≥ 27. This brings the final resolution of the conjecture by the approach of Moulin Ollagnier within range of the computationally feasible.

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Repetitions in words have been studied since the beginning of the previous century [15,16]. Recently, there has been much interest in repetitions with fractional exponent [1,3,6–8,10]. For rational r with 1 < r ≤ 2, a fractional r-power is a non-empty word w = xx′ such that x′ is the prefix of x of length (r−1)|x|. For example, 010 is a 3/2-power. A basic problem is that of identifying the repetitive threshold for each alphabet size n ≥ 1:

What is the infimum of r such that an infinite sequence on n letters exists, not containing any factor of exponent greater than r?

The infimum is called the repetitive threshold of an n-letter alphabet, denoted by RT(n). Dejean’s conjecture [6] is that

\[
RT(n) = \begin{cases} 
7/4, & n = 3 \\
7/5, & n = 4 \\
n/(n−1), & n \neq 3, 4.
\end{cases}
\]


Recently, Carpi [3] showed that Dejean’s conjecture holds for n ≥ 33. Carpi’s result is computation-free, and resolving Dejean’s conjecture is thus reduced to filling a finite gap. Conceptually, one would hope that the gap could now be filled

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1 Department of Mathematics and Statistics, University of Winnipeg, 515 Portage Avenue, Winnipeg, Manitoba R3B 2E9, Canada; j.currie@uwinnipeg.ca

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from below, using the methods of [11,12]. Since these approaches are computationally intensive, optimizing Carpi’s result is important. The present authors improved part of Carpi’s constructions to show that Dejean’s conjecture holds for $n \geq 30$ (see [4]). In the present note we show that in fact Dejean’s conjecture holds for $n \geq 27$.

**Remark 1.** Some months after the first draft of this paper, its goal has been vindicated: the final resolution of the conjecture via methods of Moulin Ollagnier becomes computationally feasible; in a recent paper the present authors proved Dejean’s conjecture by resolving computationally the cases $n \leq 26$. Dejean’s conjecture is correct! (See [5] and also [14] for another independent proof.)

The following definitions are from [3]: for any non-negative integer $r$, let $A_r = \{1, 2, \ldots, r\}$. Fix $n \geq 27$. Let $m = \lfloor (n - 3)/6 \rfloor$. If $a$ is a letter, then let $|v|_a$ denote the number of occurrences of $a$ in the word $v$. Let $\psi = \{v \in A_m^* \mid \forall a \in A_m, 4$ divides $|v|_a\}$. (We use this as a definition; it is in fact the assertion of Carpi’s Lem. 9.1.) A word $v \in A_m^*$ is a $\psi$-kernel repetition if it has period $q$ and a prefix $v'$ of length $q$ such that $v' \in \ker \psi$ and $(n - 1)(|v| + 1) \geq nq - 3$. In [4] we introduced the following definition: if $v$ has period $q$ and its prefix $v'$ of length $q$ is in $\ker \psi$, we say that $q$ is a kernel period of $v$.

Let $B = \{0, 1\}$ and let $S_n$ be the permutation group on $n$ elements. Consider the morphism $\phi : B^* \to S_n$ generated by

$$
\begin{align*}
\phi(0) &= (1 \ 2 \ 3 \ \cdots \ (n - 1)) \\
\phi(1) &= (1 \ 2 \ 3 \ \cdots \ (n - 1) \ n).
\end{align*}
$$

This map is due to Pansiot [13]. A word $u \in B^*$ is a $k$-stabilizing word if $\phi(u)$ fixes $\{1, 2, 3, \ldots, k\}$. The set of $k$-stabilizing words (for fixed $n$) is denoted by $\text{Stab}_n(k)$.

A map $\gamma_n : B^* \to A_m^*$ is defined by

$$
\gamma_n(b_1b_2\ldots b_\ell) = a_1a_2\ldots a_\ell
$$

where $a_1, \phi(b_1b_2\ldots b_\ell) = 1$ for $1 \leq i \leq \ell$.

Carpi introduces a morphism $f : A_m^* \to B^*$ generated by

$$
\begin{align*}
f(1) &= y^p x(101)^{2m} \\
f(a) &= y^p x(101)^{2m-2a}010(101)^{2a-1}
\end{align*}
$$

where $2 \leq a \leq m$, $p = \lfloor n/2 \rfloor$, $y$ is the suffix of $(01)^n$ of length $n - 1$ and $x$ is the suffix of $y$ of length $|y| - 6m$.

The concepts of so-called short repetitions and kernel repetitions were introduced by Moulin Ollagnier [12]. His work is complicated by the fact that his short repetitions are words over $A_m$, while his kernel repetitions are words over $B$ (although they code words over $A_n$ via Pansiot’s map). Without going into the details, we recall that he reduced the construction of an infinite word over $n$ letters
attaining threshold \( n/(n-1) \) to avoiding both short repetitions and kernel repetitions. Moulin Ollagnier’s binary words were fixed points of morphisms. In [11], a technique was introduced for dealing separately with short repetitions and kernel repetitions; the binary words given there can be viewed as being produced by HD0L’s: they have the form \( g(h^{\omega}(0)) \) where all words coded by \( g(B^*) \) avoid short repetitions, and each \( h \) is chosen to eliminate kernel repetitions.

Carpi’s work follows essentially this strategy. The lemmas of his paper show that \( f(B^*) \) avoids short repetitions if \( n \geq 30 \). For \( m = 5 \) (corresponding to \( n \geq 33 \)) he produces an infinite word \( w_{5} \) over \( A_{m} \) such that \( f(w_{5}) \) avoids kernel repetitions. The exact statement of this division of work into short vs. kernel repetitions is the following:

**Proposition 2** ([3], Prop. 3.2). Let \( v \in B^* \). If a factor of \( \gamma_{n}(v) \) has exponent larger than \( n/(n-1) \), then \( v \) has a factor \( u \) satisfying one of the following conditions:

(i) \( u \in \text{Stab}_{n}(k) \) and \( 0 < |u| < k(n-1) \) for some \( k \leq n-1 \);

(ii) \( u \) is a kernel repetition of order \( n \).

In our previous note, we improved only the second part of Carpi’s construction; he had shown that for \( n \geq 30 \), no factor \( u \) of \( f(A_{m}^{*}) \) satisfied condition (i) above. As Carpi therefore states at the beginning of Section 9 of [3]:

By the results of the previous sections, at least in the case \( n \geq 30 \), in order to construct an infinite word on \( n \) letters avoiding factors of any exponent larger than \( n/(n-1) \), it is sufficient to find an infinite word \( w \) on the alphabet \( A_{m} \) avoiding \( \psi \)-kernel repetitions.

For \( m = 5 \), Carpi was able to produce such an infinite word, based on a paper-folding construction. He thus established Dejean’s conjecture for \( n \geq 33 \). The present authors refined this by constructing an infinite word \( w_{4} \) on the alphabet \( A_{4} \) avoiding \( \psi \)-kernel repetitions. This established Dejean’s conjecture for \( n \geq 30 \). We remark that for \( 30 \leq n \leq 32 \) the word on \( A_{n} \) verifying Dejean’s conjecture for \( n \) is \( \gamma_{n}(v) \), where \( v = f(w_{4}) \).

In the present note, we improve on the first aspect of Carpi’s attack, by showing that for \( 27 \leq n \leq 29 \), no factor \( u \) of \( v = f(w_{4}) \) satisfies (i) above. This implies that Dejean’s conjecture holds for \( n \geq 27 \). Since \( f \) is \( r \)-uniform where \( r = (p+1)(n-1) \), to show that (i) holds for \( v \) it suffices to check that no factor \( u \in f(B^{3}) \) satisfies (i). In principle, this involves considering all factors of \( f(B^{3}) \) of length less than \( (n-1)^{2} \). However, we shorten this computation considerably by considering several of Carpi’s lemmas.

**Lemma 3.** Suppose \( n \geq 18 \). Suppose that \( u \in f(A_{m}^{*}) \setminus \text{Stab}_{n}(k) \) and \( |u| < k(n-1) \) for some \( k \in \{1, 2, \ldots, n-1\} \). Then \( |u| = r(n-1) \) for some \( r, p+1 \leq r < k \leq 16 \).

**Proof.** Propositions and lemmas referenced in this proof are in [3]. By Proposition 5.1, \( k \geq 4 \) so that \( u \in \text{Stab}_{n}(4) \). It then follows from Proposition 6.3 that \( |u| \geq (p+1)(n-1) \). Since \( |u| < k(n-1) \), we deduce that \( k > p+1 \). From \( n \geq 18 \) this means that \( k > 10 \), so that surely \( u \in \text{Stab}_{n}(7) \). Applying Lemma 7.1, we see
that $|u|$ is divisible by $n - 1$. We may thus write $|u| = r(n - 1)$, $p + 1 \leq r < k$. By the contrapositive of Proposition 7.2, $u \not\in \text{Stab}_n(17)$. It follows that $k \leq 16$. □

We verify that Dejean’s conjecture holds for $n = 27, 28, 29$ by exhaustively examining factors $u$ of $f(B_3)$ of length $r(n - 1)$ for $p + 1 \leq r \leq 15$, and verifying that such $u$ are not in $\text{Stab}_n(k)$ for any $k$, $r < k \leq 16$. For $n = 28, 29$, the check only involves $r = 15$, $k = 16$. For $n = 27$, we also must consider $r = 14$. Code written in SAGE running on a PC performed the necessary verifications in about half an hour. The code is available at www.uwinnipeg.ca/~currie/kstab.sage.

References


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