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Neighbourhoods, Classes and Near Sets

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Abstract

The article calls attention to the relationship between neighbourhoods and tolerance classes in the foundations of tolerance near sets. A particular form of tolerance relation is given by way of introduction to descriptively ε-near sets. Neighbourhoods and tolerance classes have practical applications in digital image analysis.

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1 Introduction

The paper distinguishes between neighbourhoods and tolerance classes by way of introduction to the foundations for tolerance near sets. This work is directly related to recent work on near sets, in general, and tolerance near sets, in particular (see, *e.g.*, [1, 2, 4, 7, 12]) as well as recent work on distance functions called merotopies and approach spaces (see, *e.g.* [10]).

2 Preliminaries

To introduce near set theory it is necessary to establish a basis for describing elements of sets. All sets in near set theory consist of perceptual objects.

Definition 1. Perceptual Object. A *perceptual object* is something perceivable that has its origin in the physical world.

A perceptual object is anything in the physical world with characteristics observable to the senses such that they can be measured and are knowable to the mind. Examples of perceptual objects include patients, components belonging to a manufacturing process, and camera images. Here, the term *perception* is considered relative to measurable characteristics called the object's features.

In keeping with the approach to pattern recognition suggested by M. Pavel [5], the features of an object are quantified by probe functions.

Definition 2. Probe Function [6]. A *probe function* is a real-valued function representing a feature of a perceptual object.

In this work, probe functions are defined in terms of digital images such as: colour, texture, contour, spatial orientation, and length of line segments along a bounded region. Specifically, objects in our visual field are viewed in terms of feature vectors that consist of extracted probe function values. Selected probe functions are used to measure characteristics of visual objects and provide a basis for detecting similarities among perceived objects. Put $\vec{\phi}(x) = (\phi_1(x), ..., \phi_n(x))$ for a feature vector in $\mathbb{R}^n$ that describes an object x. In sum, probe functions make it possible to determine if two objects are associated with the same pattern without necessarily specifying which pattern (as is the case when performing classification).

Next, a perceptual system is a set of perceptual objects, together with a set of probe functions.

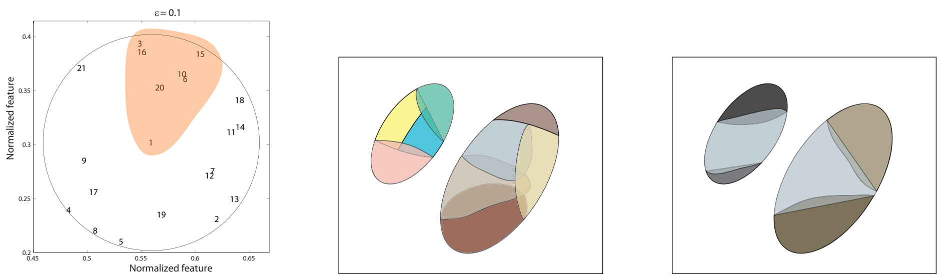
Definition 3. Perceptual System [7]. A *perceptual system* $\langle O, \mathbb{F} \rangle$ consists of a nonempty set O of sample perceptual objects and a nonempty set $\mathbb{F}$ of real-valued functions $\phi \in \mathbb{F}$ such that $\phi : O \to \mathbb{R}$.

3 Neighbourhoods and Tolerance Classes

Definition 4. Neighbourhood. Let $\langle O, \mathbb{F} \rangle$ be a perceptual system, $x \in O$ and let $\rho : O \times O \to \mathbb{R}$ denote a distance function, *e.g.*, $\rho(x, y) = |x - y|$ (standard distance). For a set $\mathcal{B} \subseteq \mathbb{F}$ and $\varepsilon \in [0, \infty]$, a closed *neighbourhood* of x (denoted by $N(x)$) is defined by

$$N_\rho(x, \varepsilon) = \{y \in O : \rho(x, y) = |x - y| \leq \varepsilon\},$$

with centre x and radius ε. For simplicity when the radius and distance function ρ are understood, we write $N(x)$. An example of a neighbourhood (also concisely written *nbd*) in a 2D feature space is given in Fig. 1.1, where the positions of the objects are given by the numbers 1 to 21, and the nbd is defined with respect to the object labelled 1. That is, the nbd is the set of all the objects within the circle. Notice that the distance between all the objects and object 1 in Fig. 1.1 is less than or equal to $\epsilon = 0.1$, and that the distances for all pairs of objects in the nbd of $x = 1$ are not necessarily less than or equal to $\epsilon = 0.1$.



1.1: Nbd & tolerance class in 2D space

1.2: Coverings with no correspondence

1.3: Covering using Def. 8

Figure 1: Visualisation of Neighbourhoods & Tolerance Classes

Definition 5. Perceptual Tolerance Relation [8, 9]. Let $\langle O, \mathbb{F}\rangle$ be a perceptual system and let $\varepsilon \in \mathbb{R}$. For every $\mathcal{B} \subseteq \mathbb{F}$, the *perceptual tolerance relation* $\cong_{\mathcal{B},\epsilon}$ is defined as follows:

$$\cong_{\mathcal{B},\varepsilon}= \{(x, y) \in O \times O :\parallel \vec{\phi}(x) - \vec{\phi}(y) \parallel\leq \varepsilon\},$$

Next, pre-classes are distinguished from tolerance classes.

Definition 6. Pre-Class. Let $\langle O, \mathbb{F}\rangle$ be a perceptual system. For $\mathcal{B} \subseteq \mathbb{F}$ and $\varepsilon \in \mathbb{R}$, a set $X \subseteq O$ is a *pre-class* iff $x \cong_{\mathcal{B},\epsilon} y$ for any pair $x, y \in X$.

Definition 7. Tolerance Class. A maximal pre-class with respect to inclusion is called a *tolerance class*.

The set coloured orange in Fig. 1.1 is a tolerance class, since no objects can be added to the set and still satisfy the condition that any pair $x, y \in$ *Orange* $\times$ *Orange* must be within ε of each other.

Theorem 1. [1] All tolerance classes containing $x \in O$ are subsets of the neighbourhood $N(x)$.

Proof. Given in [1]. □

Next, observe that objects can belong to more than one tolerance class. Consequently, the following notation is required to differentiate between classes and facilitate discussions in subsequent sections. The set of all tolerance classes using only the objects in O is given by $H_{\cong_{\mathcal{B},\varepsilon}}(O)$ (also called the cover of O), a single tolerance class is represented by $C \in H_{\cong_{\mathcal{B},\varepsilon}}(O)$, and the set of all tolerance classes containing an object x is denoted by $C_x \subset H_{\cong_{\mathcal{B},\varepsilon}}(O)$. This section concludes with another tolerance relation similar to the weak indiscernibility relation [6].

Definition 8. Weak Perceptual Tolerance Relation [8] Let $\langle O, \mathbb{F} \rangle$ be a perceptual system and let $\varepsilon \in \mathbb{R}$. For every $\mathcal{B} \subseteq \mathbb{F}$ the *weak perceptual tolerance relation* $\simeq_{\mathcal{B},\epsilon}$ is defined as follows:

$$\simeq_{\mathcal{B},\epsilon} = \{(x, y) \in O \times O : \exists \phi_i \in \mathcal{B} \centerdot \| \phi_i(x) - \phi_i(y) \|_2 \leq \varepsilon\}.$$

The weak tolerance relation can provide new information or relationships for a set of objects for a given application.

4 Tolerance Near Sets

Sets of objects that have similar descriptions are called near sets, and a method for determining similarity is provided by way of the perceptual tolerance relation (and to a lesser degree with the weak perceptual tolerance relation). The following two definitions enunciate the fundamental notion of nearness between two sets.

Definition 9. Tolerance Nearness Relation [8, 9]. Let $\langle O, \mathbb{F} \rangle$ be a perceptual system and let $X, Y \subseteq O, \varepsilon \in \mathbb{R}$. A set X is near to a set Y within the perceptual system $\langle O, \mathbb{F} \rangle$ ($X \underline{\bowtie}_{\mathbb{F}} Y$) iff there exists $x \in X$ and $y \in Y$ and there is $\mathcal{B} \subseteq \mathbb{F}$ such that $x \cong_{\mathcal{B},\epsilon} y$.

Definition 10. Tolerance Near Sets [8, 9]. Let $\langle O, \mathbb{F} \rangle$ be a perceptual system and let $\varepsilon \in \mathbb{R}, \mathcal{B} \subseteq \mathbb{F}$. Further, let $X, Y \subseteq O$, denote disjoint sets with coverings determined by the tolerance relation $\cong_{\mathcal{B},\epsilon}$, and let $H_{\cong_{\mathcal{B},\varepsilon}}(X), H_{\cong_{\mathcal{B},\varepsilon}}(Y)$ denote the set of tolerance classes for X, Y, respectively. Sets X, Y are *tolerance near sets* iff there are tolerance classes $A \in H_{\cong_{\mathcal{B},\varepsilon}}(X), B \in H_{\cong_{\mathcal{B},\varepsilon}}(Y)$ such that $A \underline{\bowtie}_{\mathbb{F}} B$.

Defs. 9 and 10 can be summarised in a single theorem.

Theorem 2. The following assertions are equivalent.

1. X, Y are tolerance near sets,
2. There are $A \in H_{\cong_{\mathcal{B},\varepsilon}}(X), B \in H_{\cong_{\mathcal{B},\varepsilon}}(Y)$ such that $A \underline{\bowtie}_{\mathbb{F}} B$,
3. There are $x \in X$, $y \in Y$, $\mathcal{B} \subseteq \mathbb{F}$ such that $x \cong_{\mathcal{B},\epsilon} y$.

Proof. Immediate from Def. 9 and Def. 10. □

Definition 9 fits nicely in applications that emphasise objects (see, *e.g.*, [1, 2]), while Definition 10 works best in applications that focus on tolerance classes. Also, notice that tolerance near sets are a variation of the original near sets using the indiscernibility relation [6]. Moreover, the original definition of tolerance near sets given in [8, 9] defines nearness in terms of pre-classes (as opposed to tolerance classes as given in Definition 10), however the results presented in [1] are obtained using tolerance classes, and so the definition was adjusted accordingly. Finally, an example of tolerance near sets is given in Fig. 1.3, where the colours represent different tolerance classes, and classes with the same colour represent the situation where $A \underline{\bowtie}_{\mathbb{F}} B$. By contrast, the ovals in Fig. 1.2 are not tolerance near sets.

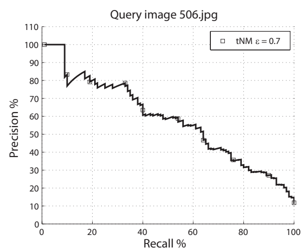


Figure 2: Precision vs. recall plot using tNM to perform CBIR on the SIMPLcity image database with query image 506 [11]

3.1: 3.2: 3.3: 3.4: 3.5:

Figure 3: Top 5 query results (query image left).

5 Application Using Near System

A direct application of Defs. 7, 9, & 10, by way of the nearness measure tNM (see [1]), is in the area of Content-Based Image Retrieval (CBIR). In this problem domain, the goal is to retrieve all the images that are similar to a query image based on the content of the image (rather than retrieval based on the strings associated with the image). To perform CBIR, a nearness measure is used to quantify the number of tolerance classes shared between disjoint sets obtained from two different images. This measure can then be used to rank all the images in a databased based on the query image. Finally, if the database is partitioned into categories, precision vs. recall plots can be used to demonstrate the ability of this approach to correctly retrieve images belonging to the same category as the query image. This approach was implemented in the near system, introduced in [3], and the results are given in the following Fig. 2 & 3.

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