THE UNIVERSITY OF CALGARY

NON REPETITIVE WALKS IN GRAPHS AND DIGRAPHS

by

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Abstract

A word w over alphabet Σ is non-repetitive if we cannot write w = abbc; a, b, c $\in \Sigma^*$, b $\neq \epsilon$. That is, no subword of w appears twice in a row in w. In 1906, Axel Thue, the Norwegian number theorist, showed that arbitrarily long non-repetitive words exist on a three letter alphabet.

Call graph or digraph G versatile if arbitrarily long non-repetitive words can be walked on G. This work deals with two questions:

(1) Which graphs are versatile?

(2) Which digraphs are versatile?

Our results concerning versatility of digraphs may be considered to give information about the structure of non-repetitive words on finite alphabets.

We attack these questions as follows:

(I) We introduce a partial ordering of digraphs called mimicking. We show that if digraph G mimics digraph H, then if H is versatile, so is G.

(II) We then produce two sets of digraphs MIN and MAX, and show that every digraph of MIN is versatile (These digraphs are intended to be minimal in the mimicking partial order with respect to being versatile.) and no digraph of MAX is versatile. (The digraphs of MAX are

intended to be maximal with respect to not being versatile.)

(III) In a lengthy classification, we show that every digraph either mimics a digraph of MIN, and hence is versatile, or "reduces" to some digraph mimicked by a digraph of MAX, and hence is not versatile.

We conclude that a digraph is versatile exactly when it mimics one of the digraphs in the finite set MIN. The set MIN contains eighty-nine (89) digraphs, and the set MAX contains twenty-five (25) individual digraphs, and one infinite family of digraphs.

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Table of Contents

List of Figures

Figure | Page

Chapter 1: Introduction

Definitions and Preliminaries:

A word is a finite sequence of elements of some finite set Σ. We call the set Σ an alphabet, the elements of Σ letters. The set of all words over Σ is denoted by Σ^*, the set of words of positive length over Σ by Σ^+. We take a naive view of words as strings of letters; thus the concatenation of two words w and v, written wv, is simply the string consisting of the letters of w followed by the letters of v. Say that v is a subword of w if we can write $w = uvz$; $u, v, z \in \Sigma^*$. We say v is a prefix (suffix) of w if we can write $w = vz$ (zv); $v, z \in \Sigma^*$. The empty word, denoted ϵ, is the word with no letters in it. Denote by $|w|$ the length of w, equal to the number of letters in w.

Let Σ, Γ be alphabets. A substitution $h: \Sigma^* \to \Gamma^*$ is a function generated by its values on Σ. That is, if w is a word on Σ, $w = a_1 a_2 \ldots a_m$, $a_i \in \Sigma$, $1 \leq i \leq m$ then $h(w) = h(a_1)h(a_2)\ldots h(a_m)$.

Define a word of type ω, to be a countable sequence of letters over some alphabet Σ. If $h: \Sigma^* \to \Sigma^*$ is some substitution with b a prefix of h(b) for some $b \in \Sigma$, and h(b) longer than b, then denote by $h^{\omega}(b)$ be the word of type ω having initial segment $h^n(b)$ for every n. This limit makes sense, as $h^n(b)$ will be a prefix of $h^{n+1}(b)$ for each n.

A word w over alphabet Σ is <u>non-repetitive</u> if we cannot write w = abbc; a, b, c $\in \Sigma^*$, b $\neq \epsilon$. That is, no subword of w appears twice in a row in w. The term <u>square-free</u> is also used for such words in the literature.

For the purposes of this thesis, a <u>graph</u> (<u>digraph</u>) G = $\langle$ V, E $\rangle$ consists of a finite set V of <u>vertices</u>, together with a set E of unordered (ordered) pairs of vertices. If G is a graph, denote by vert(G) the set of vertices of G. If a, b $\in$ V, and (a, b) $\in$ E, then we say the <u>edge</u> ab is in G. An edge of the form aa, a $\in$ V, is called a <u>loop</u>. For technical reasons to become apparent later (See Lemma 3.5), we allow digraphs to contain loops. However, we only consider graphs not containing loops.

If G is a graph or digraph we may consider V = vert(G) to be an alphabet. We say that the word w $\in V^*$ is a <u>walk</u> on G if whenever ab is a two letter subword of w, then ab is an edge of G. We say that w can be walked on G, or G allows walk w. A graph or digraph G is called <u>versatile</u> if arbitrarily long non-repetitive words can be walked on G. This work deals with two questions:

(1) Which graphs are versatile?

(2) Which digraphs are versatile?

<u>Background:</u> In 1906, Axel Thue, the Norwegian number theorist, showed that arbitrarily long non-repetitive words exist on a three letter alphabet. (See [19].) This result has been rediscovered many times, by Arshon [1], Morse and Hedlund [12] and Hawkins and Mientka [10], for example.

This result of Thue is counter-intuitive, and interesting for its own sake. It is also useful for the construction of pathological objects and counterexamples. An important example of a use of Thue's result is in the solution of the Burnside problem by Novikov and Adjan [13].

There is a large literature concerning non-repetitive words (See the bibliography of Bean, Ehrenfeucht and McNulty [3].) By Konig's lemma, the existence of arbitrarily long non-repetitive words on a finite alphabet is equivalent to the existence of a non-repetitive word of type ω on that alphabet. Shelton and Soni [16], [17], [18] investigate the structure of the set of non-repetitive ω words on a three letter alphabet, showing the set to be perfect with respect to a natural metric.

Call a word w over alphabet Σ <u>strongly cube-free</u> if we cannot write $w = abb\beta c$, where $a, b, c \in \Sigma^*$, $\beta \in \Sigma$, and β is the first letter of b. If Σ is a two letter alphabet and τ a three letter alphabet, then a strongly cube-free word of type ω over Σ gives rise to a non-repetitive word of type ω over τ in a natural way, and vice versa. (See Braunholtz [5].) Fife [9] shows that the strongly cube-free words of type ω over a two letter alphabet form a Cantor set under a natural metric.

The study of words which are non-repetitive or strongly cube-free is generalized in Bean, Ehrenfeucht and McNulty [3]. Here the question of words avoiding an arbitrary pattern is considered. A word $w \in \Sigma^*$ <u>avoids</u> the word $v = b_1b_2...b_m$ if we cannot write $w = ah(b_1b_2...b_m)c$ where $a, c \in \Sigma^*$, and h is a substitution not mapping any of the b_i to the empty word. An algorithm is given to determine whether, given v, there exists a natural number n, so that there exist arbitrarily long words avoiding v on an n letter alphabet. If such an n exists, v is said to be <u>avoidable</u>. If v is avoidable, it is natural to attempt to bound the n mentioned above. This problem is attacked in the paper Baker, McNulty, Taylor [2]. From [2], the follwing question naturally arises: On which directed graphs can arbitrarily long non-repetitive words be walked?

As mentioned, this question is the subject of the present thesis. In a different light, one may consider this question to be in the spirit of the investigations of Shelton, Soni and Fife: What can we say about the structure of non-repetitive words?

Let w be a word of type ω over alphabet Σ. Baker, McNulty and Taylor define the <u>transition digraph</u> of w to be that digraph having vertex set Σ, and an edge $a_i a_j$, $a_i, a_j \in \Sigma$, exactly when $a_i a_j$ is a subword of w. It is shown in [2] that if w is a non-repetitive word of type ω on the three letter alphabet { a, b, c }, then w must have a transition digraph with edges ab, ba, ac, ca, bc, cb. Equivalently, a digraph on vertices a, b, c is versatile only if it contains the six edges ab, ac, ba, bc, ca, cb. Our results concerning versatility of digraphs may thus be considered to give information about the structure of non-repetitive words on finite alphabets.

Choffrut and Culik [7] consider the following problem: Let Σ be a finite alphabet, $w_1, w_2, \ldots, w_m$ words over Σ. Do there exist arbitrarily long words over Σ not including any of the w_i as subwords? Thus while Bean, Ehrenfeucht and McNulty consider the problem of avoiding patterns, Choffrut and Culik wish to avoid specific words. The present work may be considered a hybrid of these two approaches: If D is a digraph with vertices v_1, v_2, .., v_n, we wish if possible to find arbitrarily long words on { $v_1, v_2, \ldots, v_n$ } avoiding the pattern xx, and simultaneously avoiding the specific words $v_i v_j$, where $v_i v_j$ is any non-edge of D.

Outline: Having motivated our work in the previous section, we make some remarks concerning our attack:

(I) We introduce a partial ordering of digraphs called mimicking. We show that if digraph G mimics digraph H, then if H is versatile, so is G.

(II) We then produce two sets of digraphs MIN and MAX, and show that every digraph of MIN is versatile (These digraphs are intended to be minimal in the mimicking partial order with respect to being versatile.) and every digraph of MAX is not versatile. (The digraphs of MAX are intended to be maximal with respect to not being versatile.)

(III) In a lengthy classification, we show that every digraph either mimics a digraph of MIN, and hence is versatile, or "reduces" to some digraph mimicked by a digraph of MAX, and hence is not versatile.

We conclude that a digraph is versatile exactly when it mimics one of the digraphs in the finite set MIN.

Step (II) naturally presupposes the construction of certain non-repetitive words. From Axel Thue on down, those wishing to construct squarefree words have used substitutions. A substitution $h: \Sigma^* \to \Gamma^*$ is called <u>square-free</u> if whenever $w \in \Sigma^*$ is non-repetitive, so is $h(w)$. Axel Thue showed that the substitution $h: \{ a, b, c \}^* \to \{ a, b, c \}^*$ given by

$h(a) = abcab$

$h(b) = acabcb$

$h(c) = acbcacb$

is squarefree. It follows that $h^{\omega}(a)$ is a non-repetitive word of type ω on Σ. On the other hand, the substitution $g: \{ a, b, c \}^* \to \{ a, b, c \}^*$ given by

$g(a) = c$

$g(b) = bca$

$g(c) = ba$

is not square-free. In fact $g(bcb) = bcababca$, which contains the repetition abab. Nonetheless, the fact that $g^{\omega}(b)$ is non-repetitive was proved by Arshon [1] in the

1930's. Crochemore [8], defines a concept of <u>weak square-freeness</u> for substitutions. Let Σ be an alphabet. Then h: $\Sigma^* \to \Sigma^*$ is weakly square-free if there exist x, w, where $x \in \Sigma$, $w \in \Sigma^+$, such that h(x) = xw, and $h^\omega(x)$ is non-repetitive. Although g is not square-free, g is weakly square-free.

Let f: { 1, 2, 3 }$^* \to \Sigma^*$ be a substitution. In the body of this thesis (see Lemma 2.4.), we prove that under certain conditions on f, $f(g^\omega(b))$ is non-repetitive, with g given as above. These conditions do not force f to be square-free, in fact f(bcb), f(aca) are explicitly allowed to contain repetitions. This result is used to produce non-repetitive words of type ω. Except in one case, all of the many non-repetitive walks used in this thesis are of the form $f(g^\omega(a))$ for such an f. In the other case we generate a non-repetitive word using weakly squarefree substitutions on a five element alphabet.

Much work has been done on square-free substitutions, and cube-free substitutions, which are defined analogously. References may be found in the bibliographies of Berstel [4] and Crochemore [8]. We give an example of a particularly beautiful result of Karhumaki [11]:

Theorem: Let h: $\{a, b\}^* \to \{a, b\}^*$ be a substitution such that h(a) begins with an a. Then the word $h^\omega(a)$ is cube-free if and only if the word $h^{10}(a)$ is cube-free.

One last remark is in order, of interest to those following the work of Robertson, Seymour [15]: One might ask why we consider graphs separately from digraphs, since a graph G may be considered to be simply a symmetric digraph. It turns out that the solution of the graph case of our problem allows us to find a nice classification scheme for digraphs. Moreover, it follows from the work of Robertson, Seymour on graph minors that the graph case will have a nice solution: From the weaving lemma of chapter 2 one may deduce that if G does not allow arbitrarily long non-repetitive walks, then neither does any minor of G. Thus [15] implies that there is an excluded minor characterization of those graphs not allowing arbitrarily long non-repetitive walks. We know of no generalization of the work of[15] to digraphs.

Open Problems: (1) It was remarked above that of the digraphs on three vertices a, b, c, only a digraph including edges ab, ba, bc, cb, ca, ac allows arbitrarily long non-repetitive walks. We can show that if w is a non-repetitive word of type ω on three letters a, b, c, then w must contain as subwords all of the words in one of the following sets (up to a permutation of letters):

H_1 = { aba, abc, acb, bab, bac, bca, cab, cac, cba, cbc }

H_2 = { abc, aca, acb, bac, bca, bcb, cac, cab, cba, cbc }

A non-repetitive word of type ω all of whose three letter subwords are in H_1 is $g^{\omega}(b)$ where g is Arshon's substitution, given above.

A non-repetitive word of type ω all of whose three letter subwords are in H_2 is $g(f^{\omega}(1))$ where f,g are given by

f(1) = 142

f(2) = 1435

f(3) = 143532

f(4) = 1532

f(5) = 1535

g(1) = ac

g(2) = acb

g(3) = acbc

g(4) = abc

g(5) = abcb

That $g(f^{\omega}(1))$ is non-repetitive may be proved using the methods of Chapter 7 although this fact is not used in this thesis. In general, if w is a non-repetitive word of type ω on n letters, what k-letter subwords must w contain? (This question could be phrased in the language of hypergraphs.)

(2) Call a word w strongly non-repetitive if we cannot write w = abcd, a, b, c, d $\in \Sigma^*$, b $\neq \epsilon$, c a permutation of b. There exists a strongly non-repetitive word of type ω on a five letter alphabet. Whether such a word exists on four letters is an open problem. (See [6], [14]) On which digraphs can arbitrarily long strongly non-repetitive words be walked?

Chapter 2: Graphs.

We start this chapter with some definitions concerning graphs and digraphs.

Let G be a graph (digraph) with vertex set V, a, b $\in$ V. We say that the word $p \in (V \setminus \{ a, b \})^*$ is a (directed) path in G from a to b if the word apb is a walk in G, and no vertex of G appears in p twice. The graph P_i whose vertex set is { 1, 2, ..., i } and whose edges are 12, 23, ..., (i-1)i, is called the path on i vertices.

A graph or digraph G is connected if for every a, b $\in$ V, a $\neq$ b, there is either a path in G from a to b, or a path in G from b to a. A digraph G is strongly connected if for every a, b $\in$ V, there is a path in G from a to b and a path in G from b to a.

Let G be a graph (digraph) with vertex set V, a $\in$ V. Let $p \in (V \setminus \{ a \})^*$ be a word, $p \neq \epsilon$. If no vertex of V appears twice in p, and both ap and pa are walks in G, then we say that the word ap is a cycle of G based at a, or simply, a cycle of G. (Various terms exist in the literature. Others are circuit, and simple cycle.) A graph C whose vertices are { c_1, c_2, ..., c_m } and whose edges are c_1c_2, c_2c_3, ..., $c_{m-1}c_m$, c_mc_1 is called a cycle.

If G is a graph (digraph), $a, b \in V$, then if ab is an edge of G, say that <u>b is a neighbour of a</u> (<u>b is a successor of a</u>, <u>a is a predecessor of b</u>). The degree (indegree, outdegree) of a is the number of neighbours (predecessors, successors) of a in G.

If G_1, G_2 are graphs (digraphs) with vertex sets V_1, V_2 and edge sets E_1, E_2 then denote by $G_1 \cap G_2$ the graph with vertex set $V_1 \cap V_2$ and edge set $E_1 \cap E_2$. Analogously define $G_1 \cup G_2$.

In this chapter, we answer the question: Which graphs are versatile? We restrict our attention to connected graphs, since a word v can be walked on a graph G if and only if v can be walked on a connected component of G. We prove the following theorem:

<u>Theorem 2.1:</u> A connected graph G is versatile unless G is a path on four or fewer vertices.

The following observation proves useful.

<u>Lemma 2.1 (a) (Weaving Lemma):</u> Let $v = a_1a_2...a_r$ be a non-repetitive word, $a_1, a_2, ..., a_r \in S$, some alphabet. Let $b_1, b_2, ..., b_{r+1}$ be non-repetitive words on alphabet T, where S and T are disjoint. We permit some or all of the b_i to be empty. Then $w = b_1a_1b_2a_2...b_ra_rb_{r+1}$ is a

non-repetitive word.

<u>Proof:</u> Suppose w contains a repetition, say $w = uyyz$, $y \neq \epsilon$. Then yy contains some a_j, for otherwise yy is a subword of one of the b_i, contradicting the fact that the b_i are non-repetitive.

Now if p is a word on $S \cup T$, denote by $p_{|S}$ the word formed by deleting from p all the letters of T. Thus the above paragraph remarks that $y_{|S} \neq \epsilon$; however, $a_1 a_2 \ldots a_r = w_{|S} = u_{|S} y_{|S} y_{|S} z_{|S}$ and therefore $v = a_1 a_2 \ldots a_r$ contains a repetition, namely $y_{|S} y_{|S}$, which is a contradiction.

We thus conclude that w is a non-repetitive word. □

Let v be a word of type ω on some alphabet S, $S = \{ a_1, a_2, \ldots, a_n \}$. Let G be a graph (digraph) including S among its vertex set. Suppose that whenever $a_i a_j \in S^*$ is a subword of v there is a path $P(a_i, a_j)$ in G from a_i to a_j such that no vertex of $P(a_i, a_j)$ is in S. We say that <u>v can be walked in G modulo paths</u>. The weaving lemma will often be applied in the following way:

<u>Lemma 2.1 (b) (Second Weaving Lemma):</u> Let v be a non-repetitive word of type ω, G a graph (digraph). If v can be walked on G modulo paths, then G is versatile.

<u>Proof:</u> Pick $n > 0$. Let $b_1 b_2 \ldots b_n$ be the initial segment of v of length n. The word w where

$$w = b_1P(b_1,b_2)b_2P(b_2,b_3)b_3\ldots b_{n-1}P(b_{n-1},b_n)b_n$$

will be a non-repetitive word by the weaving lemma. By construction, w is a non-repetitive walk on G of length n or more. Thus G allows arbitrarily long non-repetitive walks.□

We now commence the proof of Theorem 2.1, proving a series of lemmas.

Lemma 2.2: Let G be a graph with a vertex v with degree(v) $\geq$ 3. Then G is versatile.

Proof: Let three neighbours of v be a, b, c. Let w be any non-repetitive word of type ω on { a, b, c }. Then w can be walked on G modulo paths, with

$P(a,b) = P(b,a) = P(b,c) = P(c,b) = P(c,a) = P(a,c) = v$

(See Figure 2.1) Thus, by the second weaving lemma, G is versatile.□

Restating Lemma 2.2, any graph which is not versatile must have the degree of every vertex being 2 or less. In the case of connected graphs, we are left with paths and cycles.

Lemma 2.3: Let $C = c_1c_2\ldots c_m$ ($m \geq 3$) be a cycle. Then C is versatile.

Proof: Again we use the second weaving lemma. Here let v be any non-repetitive word of type ω on { c_1, c_2, c_3 }. Then v can be walked on C modulo paths,

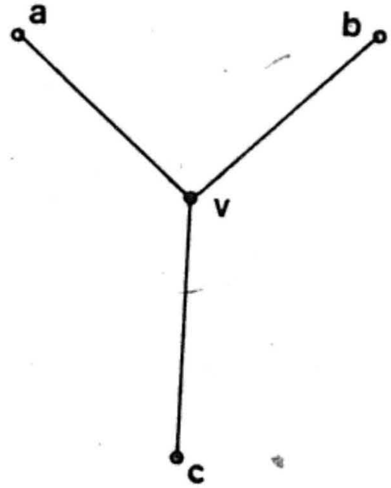


Figure 2.1

where

$$P(c_1c_2) = P(c_2c_1) = P(c_2c_3) = \epsilon P(c_3c_2) = \epsilon$$

$$P(c_3c_1) = c_4c_5\ldots c_m$$

$$P(c_1c_3) = c_mc_{m-1}\ldots c_4$$

(See Figure 2.2) Thus G is versatile.□

We have seen that every connected graph which is not a path is versatile. To conclude our examination of graphs we consider paths. Paths on four or fewer vertices do not allow arbitrarily long non-repetitive walks. It suffices to show this for P_4, since P_4 contains shorter paths as subgraphs.

Suppose that P_4 allows arbitrarily long non-repetitive walks. Then let v be a non-repetitive word of type ω which can be walked on P_4. We chop v up into blocks starting with 1. That is, consider the possible subwords of v commencing with 1, ending with 2 ,and containing exactly one 1. (See Figure 2.3). Clearly these are a = 12, b = 1232, c = 123432. However a moment's thought shows that block a cannot appear in v since the words aa, ab, ac all contain the repetition aa, and if v contains block a, then it must contain one of these longer words.

Thus v must be composed entirely of the two blocks b and c. However any non-repetitive word on two letters is

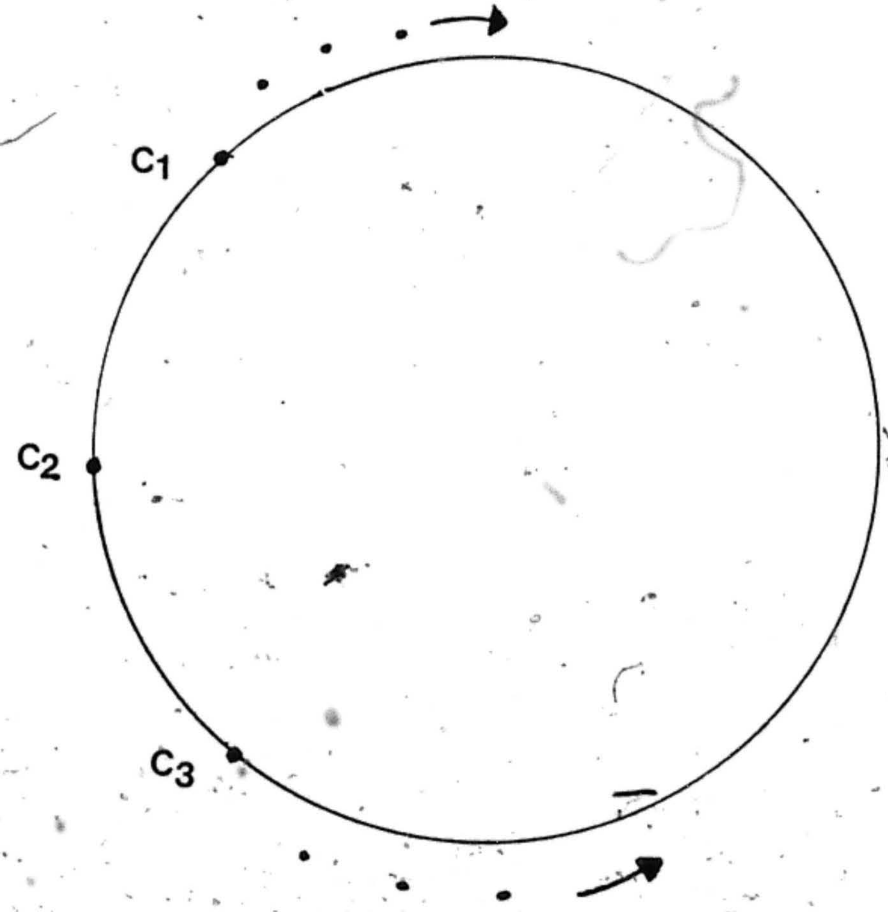


Figure 2.2

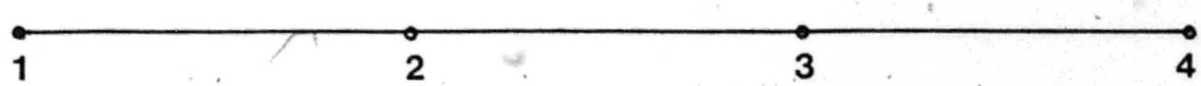


Figure 2.3

finite, hence v must be a finite word. This is a contradiction. Thus P_4 does not allow arbitrarily long non-repetitive walks.

Definition: Let $S = \langle x_1, x_2, x_3 \rangle$, T be alphabets and let $h: S^* \to T^*$ be a substitution. Say that h is suitable if

1) $|h(x_i)| \leq |h(x_j)| + |h(x_k)|$ for $1 \leq i, j, k \leq 3$, i, j, k distinct.

2) For $1 \leq i \leq 3$ one cannot write $h(x_i) = uw = wz$, $u,w,z \in T^*$, $u,w,z \neq \epsilon$.

3) If $w \in S^*$ is a non-repetitive word with $|w| = 3$ and $w \neq x_2x_3x_2, x_1x_3x_1$, then $h(w)$ is non-repetitive.

To show that P_5 allows arbitrarily long non-repetitive walks, we introduce another lemma for producing new non-repetitive words from old. In fact this lemma will be one of the main tools of this thesis.

Lemma 2.4 (Substitution Lemma): Let S be the alphabet $\{ x_1, x_2, x_3 \}$. Let $v \in S^*$ be a non-repetitive word, such that $x_2x_3x_2$, $x_1x_3x_1$ are not subwords of v. If h is suitable, then h(v) is non-repetitive.

Proof: Suppose v fulfills the conditions of the lemma and h is suitable. Let $v = a_1a_2...a_m$. For each i, $1 \leq i \leq m$, say $h(a_i) = e_i$. For the sake of a contradiction, suppose $h(v) = e_1 e_2 \ldots e_m = abbc$, some $a,b,c \in T^*$, $b \neq \epsilon$.

Without loss of generality, shortening v if necessary, write

$$e''_1 e_2 \ldots e_{j-1} e'_j = e''_j e_{j+1} \ldots e_{m-1} e'_m = b \ \ldots \ (*)$$

where $e_1 = e'_1 e''_1$, $e_j = e'_j e''_j$, $e_m = e'_m e''_m$

$e''_1, e'_j, e'_m \neq \epsilon$.

Since $h(v)$ is repetitive, $m > 3$. Otherwise, by condition 3) of the definition of suitability, $e_1 e_2 e_3$ is $x_1 x_3 x_1$ or $x_2 x_3 x_2$, contrary to our assumptions on v. Also $j > 1$, otherwise

$| e_1 | > | e_2 | + | e_3 |$ by line (*) and the fact that $m > 3$. Similarly, $j < m$.

<u>Claim:</u> The two expressions $e''_1 e_2 \ldots e'_j$ and $e''_j e_{j+1} \ldots e'_m$ "match up" in the natural way, i.e.

$e'_j = e'_m$,

$e''_1 = e''_j$,

$m = 2j - 1$ and

$e_{1+i} = e_{j+i}$ for $i = 1$ to $j-2$.

<u>Proof of Claim:</u> If $e'_j \neq e'_m$ suppose that $| e'_m | > | e'_j |$. Say that $e'_m = e''_k e_{k+1} \ldots e'_j$ for some $k < j$, and $e_k = e'_k e''_k$, $e''_k \neq \epsilon$. Then $h(a_k a_m) = e_k e_m = e'_k e''_k e'_k e_{k+1} e_{k+2} \ldots e'_j e''_m$ which contains the repetition $e''_k e''_k$. By condition 3) on h we must have $a_k = a_m$. However now

$e'_k e''_k = e_k = e_m = e'_m e''_m = e''_k e_{k+1} \ldots e'_j e''_m$. Note that $e'_j \neq \epsilon$, so that condition 2) is contradicted for $h(a_k)$, which commences and ends with e''_k.

We get a similar contradiction if $| e'_m | < | e'_j |$. Thus $e'_m = e'_j$ and $e''_1 \ldots e_{j-1} = e''_j \ldots e_{m-1}$. Repeating this argument we show that

$e_{1+i} = e_{j+i}$ for $i = 1$ to $j-2$.

$e''_j = e''_1$, and $2j - 1 = m$, as desired.□

Note that $e_{1+i} = e_{j+i}$ implies that $a_{1+i} = a_{j+i}$, since h is suitable. From the claim,

$$h(a_1 a_j a_m) = e_1 e_j e_m = e'_1 e''_1 e'_j e''_j e'_m e''_m$$
$$= e'_1 e''_1 e'_j e''_1 e'_j e''_m$$

which repeats $e''_1 e'_j$. Since $| a_1 a_j a_m | = 3$, one of the following cases must arise:

A: $a_1 = a_j$

B: $a_j = a_m$

C: $a_1 = x_1$, $a_j = x_3$, $a_m = x_1$

D: $a_1 = x_2$, $a_j = x_3$, $a_m = x_2$.

In case A, v contains the subword $a_1 a_2 \ldots a_{j-1} a_1 a_2 \ldots a_{j-1}$, which is a contradiction , as v is non-repetitive. Similarly case B cannot occur, as v would contain a repetition.

Suppose case C occurs. (Case D is similar.) Since

$m \geq 4$, and m is odd, $m \geq 5$. Therefore $j \geq 3$.

Now $a_2 = a_{j+1}$. But since v is non-repetitive, $a_2 \neq a_1 = x_1$ and $a_{j+1} \neq a_j = x_3$. Thus $a_2 = a_{j+1} = x_2$.

Also $a_{j-1} = a_{m-1}$ so that $x_3 = a_j \neq a_{j-1}$ and $a_{m-1} \neq a_m = x_1$. We conclude that $a_{j-1} = a_{m-1} = x_2$. Therefore $a_{j-1}a_ja_{j+1} = x_2x_3x_2$, contradicting our assumptions on v.

The assumption that h(v) repeats leads to a contradiction. Therefore h(v) contains no repetition.□

<u>Remarks</u>: Several variations have been proved of a lemma with stronger conditions than the above, and a stronger conclusion. For example, in Bean, Ehrenfeucht and McNulty [3], the following lemma is proved.

<u>Lemma 2.5:</u> Let Σ, Γ be alphabets. Suppose that $h: \Sigma^* \to \Gamma^*$ is a substitution such that

1') If x, y $\in \Sigma$ and h(y) is a subword of h(x), then y = x.

3') If $w \in \Sigma^*$ is a non-repetitive word with $|\, w \,| = 3$ then h(w) is non-repetitive.

Then if $v \in \Sigma^*$ is non-repetitive, so is h(v).

The proof is essentially that of Lemma 2.4, with condition 1') sufficing to prove the claim. In fact our claim, with slight renaming, comes from [3]. We have stated this result of [3] as a lemma, as we will refer to it later.

Lemma 2.4, in comparison with Lemma 2.5, restricts h less, and v more. When Σ = S, condition 3') necessitates the checking of h(w) for twelve three letter words w, whereas condition (3 only requires good behaviour from h on ten of these twelve triples.

Next we show how to produce arbitrarily long words v on S, S = { x_1, x_2, x_3 } satisfying the conditions of the substitution lemma. Consider the substitution

h: $S^* \rightarrow S^*$ where

$$h(x_1) = x_3$$
$$h(x_2) = x_2x_3x_1 \qquad \text{(Sub 2.1)}$$
$$h(x_3) = x_2x_1$$

Clearly h meets conditions 1) and 2) of the definition of suitability. (We point out that h does <u>not</u> meet condition 1' above.) That h also meets condition 3) of suitability is verified by checking the action of h on triples of S.

$$h(x_1x_2x_1) = x_3x_2x_3x_1x_3$$
$$h(x_1x_2x_3) = x_3x_2x_3x_1x_2x_1$$

$h(x_1x_3x_1) = x_3x_2x_1x_3$

$h(x_1x_3x_2) = x_3x_2x_1x_2x_3x_1$

$h(x_2x_1x_2) = x_2x_3x_1x_3x_2x_3x_1$

$h(x_2x_1x_3) = x_2x_3x_1x_3x_2x_1$

$h(x_2x_3x_1) = x_2x_3x_1x_2x_1x_3$

$h(x_2x_3x_2) = x_2x_3x_1x_2x_1x_2x_3x_1$

$h(x_3x_1x_2) = x_2x_1x_3x_2x_3x_1$

$h(x_3x_1x_3) = x_2x_1x_3x_2x_1$

$h(x_3x_2x_1) = x_2x_1x_2x_3x_1x_3$

$h(x_3x_2x_3) = x_2x_1x_2x_3x_1x_2x_1$

Only $h(x_2x_3x_2)$ contains a repetition: $x_1x_2x_1x_2$.

Let v be any non-repetitive word on S. Any x_3 appearing internally in h(v) either comes from h(x_2) and appears in the context $x_2x_3x_1$, or comes from h(x_1) and appears in the context $x_1x_3x_2$. Thus the words $x_1x_3x_1$ and $x_2x_3x_2$ are not subwords of h(v).

Now suppose $v \in S^*$ has no repetition and doesn't contain $x_1x_3x_1$ or $x_2x_3x_2$ as subwords. By the substitution lemma, h(v) contains no repetition. By our last observation, h(v) contains neither $x_1x_3x_1$ nor $x_2x_3x_2$. Thus by induction $h^n(x_2)$ has no repetitions, and does not contain $x_1x_3x_1$ or $x_2x_3x_2$. We therefore see that the word $h^n(x_2)$ fulfills the substitution lemma's conditions on v, and can be made arbitrarily long.

We are now ready to show that P_5 allows arbitrarily long non-repetitive walks. Consider the following substitution.

$g: S^* \to T^*$

$g(x_1) = 12345432$

$g(x_2) = 1234323454321234543234 32$ (Sub 2.2)

$g(x_3) = 1234323454323432$

Clearly $g(v)$ is a walk on P_5 whenever $v \in S^*$. (See Figure 2.4)

Further, g is suitable. The only condition difficult to check is condition 3). One must check these words for non-repetitiveness:

$g(x_1x_2x_1)$ = 123454321234323454321234543234321234 5432

$g(x_1x_2x_3)$ = 123454321234323454321234543234 32-
1234323454323432

$g(x_1x_3x_1)$ = 1234543212343234543234321234 5432

$g(x_1x_3x_2)$ = 123454321234323454323432123432345432-
123454323432

$g(x_2x_1x_2)$ = 12343234543212345432343212345432-
1234323454321234543234 32

$g(x_2x_1x_3)$ = 12343234543212345432343212345432-
1234323454323432

$g(x_2x_3x_1)$ = 123432345432123454323432-
12343234543234321234 5432

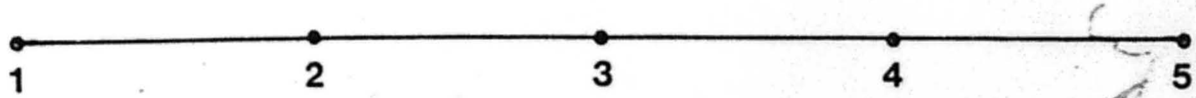


Figure 2.4

$g(x_2x_3x_2)$ = 12343234543212345432343 2-

12343234543234321234323454321234543234 32

$g(x_3x_1x_2)$ = 1234323454323432123454321234323454 32-

123454323432

$g(x_3x_1x_3)$ = 1234323454323432123454321234323454323432

$g(x_3x_2x_1)$ = 123432345432343212343234543 2-

12345432343212345432

$g(x_3x_2x_3)$ = 123432345432343212343234543 2-

1234543234321234323454323432

As an example, we show that $w = g(x_1x_2x_1)$ is non-repetitive. Suppose not. Then w must contain a repetition vv. Being a repetition, vv contains the symbol 1 exactly four, two or no times. We can rule out vv containing no 1's, since then vv would be entirely contained in one of $g(x_1)$, $g(x_2)$, $g(x_3)$, which can each be checked to be non-repetitive.

If vv contains exactly four 1's, then the first and third 1's of w are " matched " by vv:

$$\underline{\hat{1}2345432}123432345432\underline{\hat{1}2345432}343212345432$$

However, as indicated in the above scheme, this cannot happen, as the subwords of w commencing at the first and third 1's don't agree for long enough. (The extent of their agreement is underlined.)

Suppose vv contains then exactly two 1's. If the first 1 of w is contained in vv, it must be matched with the second 1 of w:

$\underline{\hat{1}234}5432\underline{\hat{1}234}32345432123454323432123454 32$

We see that this is impossible.

Suppose that vv matches the second and third 1's of w.

$1\underline{2345432\hat{1}234}3\underline{2345432\hat{1}234}54323432123454 32$

Again we see that this is impossible; the underlined "zones" of agreement for these two 1's do not meet.

The second 1 of w cannot be matched with the fourth 1, since then vv would also contain the third 1. However then vv would contain all four 1's, which is impossible, as mentioned.

The final possibility is that the third and fourth 1's of w should match. However, we note that w1 is a palindrome. Since the second and third 1's could not match, neither can the third and fourth.

By arguments of this type, all the listed words except for $g(x_2x_3x_2)$ can be shown to be non-repetitive. Alternatively, g can be shown to be suitable by invoking the Long/Short Lemma of Chapter 7.

As g is suitable, $g(\ h^n(\ x_2\ )\ )$ gives an arbitrarily long non-repetitive walk on P_5 by choosing n as large as desired. Thus P_5 is versatile. Since any path on more than five vertices contains P_5 as a subgraph, such paths are also versatile. We have thus proved Theorem 2.5.

Versatility of MIN.1 - MIN.4: Since we have the substitutions h and g handy, this is a convenient point in the thesis at which to show that MIN.1 - MIN.4 are versatile digraphs. Let $v = h^{\omega}(x_2)$, $w = g(h^{\omega}(x_2))$.

Recall from Chapter 1 the concept of a transition digraph: given a word u of type ω over a finite alphabet Σ, the transition digraph of u has as vertices those letters of Σ appearing in u, and a directed edge from letter x to letter y exactly when xy is a subword of u. Thus MIN.1 is isomorphic to the transition digraph of v, and MIN.3 is precisely the transition digraph of w. It follows that MIN.1 and MIN.3 are versatile digraphs.

As we remarked earlier, v does not contain subwords $x_2x_3x_2$ or $x_1x_3x_1$. Whenever x_3 occurs in v it is either in the context $x_2x_3x_1$ or $x_1x_3x_2$. Let v' be the word of type ω arising from v by replacing x_3 by x_4 whenever x_3 occurs in context $x_1x_3x_2$. Clearly v' will be a non-repetitive word of type ω. One checks that MIN.2 is isomorphic to the transition digraph of v', and thus is versatile.

Similarly, one checks that w does not contain subwords 232 or 434. Whenever 3 occurs in w it is either in the context 234 or 432. Let w' be the word of type ω arising from w by replacing 3 by 3' whenever 3 occurs in context 432. Again w' will be a non-repetitive word of type ω. One checks that MIN.4 is the transition digraph of w', and thus is versatile.

Chapter 3: Digraph Classification

In this chapter we ask the following question: Which digraphs are versatile? In analogy to chapter 2, we are only interested in strongly connected digraphs.

Lemma 3.1: Let G be a digraph. Then G is versatile if and only if one of G's strongly connected components is versatile.

Proof: Clearly if a component of G is versatile, so is G. Suppose that G is versatile. Let v be a non-repetitive word of type ω which can be walked on G.

We show that whenever x and y are vertices in different components of G then a final segment of v can be walked in one of $G \setminus \{ x \}$ or $G \setminus \{ y \}$. It will follow by induction on the size of G that a non-repetitive walk of type ω exists in one of G's components.

Suppose then that x and y are vertices of G and there is no directed xy path in G. If v contains no x, then v can be walked in $G \setminus \{ x \}$ and we are done. If v contains an x, then a final segment v' of v contains no y, and v' can be walked in $G \setminus \{ y \}$.□

A strongly connected digraph can be written as a union of cycles. In the following lemma we relate the intersection of these cycles to the existence of non-repetitive walks.

<u>Lemma 3.2 (Intersection Lemma):</u> Let X, Y be directed cycles in the digraph G so that vert(X) $\cap$ vert(Y) $\neq \phi$. Then either

1) X $\cap$ Y is connected

or 2) X $\cup$ Y is versatile.

<u>Proof:</u> In fact if 1) does not hold, then X U Y "contains" one of the versatile digraphs MIN 1 or MIN 2, in a sense to be made precise later. We show that $\neg$1) $\rightarrow$ 2). First note that X (similarly Y) gives a circular order to the vertices of vert(X) $\cap$ vert(Y).

<u>Case A:</u> The circular orders given to vert(X) $\cap$ vert(Y) by X and Y are different.

In this case there are vertices x_1, x_2, x_3 of X $\cap$ Y occurring in the order x_1, x_2, x_3 in the cycle X, and in the order x_1, x_3, x_2 in the cycle Y. Now we use the Second Weaving Lemma, Lemma 2.1(b). As in the last part of Chapter 2, let v be $h^{\omega}(x_2)$. The Second Weaving Lemma requires us to walk v on X $\cap$ Y modulo paths. We let the paths P(x_1, x_2), P(x_2, x_3), P(x_3, x_1) be arcs in cycle X. We require that none of these paths contain x_1, x_2 or x_3. However, this is fulfilled because of the assumed circular order of these vertices in X. For example, the vertex x_3 cannot be on the arc of X between

x_1 and x_2. The required paths P(x_1, x_3), P(x_3, x_2), P(x_2, x_1) are chosen in Y. Then v can be walked on X ∩ Y modulo these paths, and by the Second Weaving Lemma, X ∩ Y is versatile.

Case B: The circular orders on vert(X) ∩ vert(Y) given by X and Y are the same. Suppose that X ∩ Y is not connected. Then choose vertices x_1, x_2 which are in different components of X ∩ Y. Let P_X(x_1, x_2) be the x_1x_2 path in X, P_Y(x_1, x_2) the x_1x_2 path in Y. Since these two paths are not equal, we have

$$\text{vert}(\ P_X(\ x_1,\ x_2\ )\ ) \neq \text{vert}(\ P_Y(\ x_1,\ x_2\ )\ ).$$

Let $x_4 \in$ vert(P_X(x_1, x_2)) ⊕ vert(P_Y(x_1, x_2)). Using similar definitions, let $x_3 \in$ vert(P_X(x_2, x_1)) ⊕ vert(P_Y(x_2, x_1)).

We again wish to apply the Second Weaving Lemma, Lemma 2.1(b), with S = { x_1, x_2, x_3, x_4 }. Instead of v, we use v', the word arising from v by replacing x_3 by x_4 wherever x_3 occurs in context $x_1x_3x_2$.

As remarked at the end of Chapter 2 v' is non-repetitive. Also the only two letter subwords of v' are x_1x_2, x_2x_1, x_1x_4, x_2x_3, x_4x_2, x_3x_1. We must now show that we can walk v' in X ∪ Y modulo paths. There exists an x_1x_2 path in X ∪ Y not through x_4, since x_4 is not on both P_X(x_1, x_2) and P_Y(x_1, x_2). Also x_3 is not on

$P_X(x_1, x_2)$ or $P_Y(x_1, x_2)$, because x_3 is between x_2 and x_1 on one of X and Y. We may thus choose one of $P_X(x_1, x_2)$ or $P_Y(x_1, x_2)$ to serve as a path $P(x_1, x_2)$ having no vertex in S.

Preparing to use the second weaving lemma, with $S = \{x_1, x_2, x_3, x_4\}$, we have shown that the required path $P(x_1, x_2)$ exists. Further, since x_4 is between x_1 and x_2 on one of X and Y, there is an x_1x_4 path in $X \cup Y$ not through x_2. Again x_3 is not on this path, for otherwise x_3 is between x_1 and x_4, hence x_1 and x_2. Arguing similarly, the existence of paths $P(x_1, x_2)$, $P(x_1, x_4)$, $P(x_4, x_2)$, $P(x_2, d)$, $P(x_2, x_1)$, $P(x_3, x_1)$ may be shown. We may thus walk v' on $X \cup Y$ modulo paths and therefore $X \cup Y$ is versatile.

We have shown that certain digraphs are versatile. We use this intersection lemma to delineate the digraphs requiring further investigation.

<u>Lemma 3.3 (Classification Lemma)</u>: Let G be a strongly connected digraph . Then G is of one of the following types:

(1) vert(G) = vert(X) for some directed cycle X of G. In this case, say G is a <u>one hump</u> digraph.

(2) G is not of type (1), but vert(G) =

vert (X ∪ Y) where X and Y are directed cycles, and $X \cap Y$ is connected and non-empty. In this case, say G is a <u>two hump</u> digraph.

(3) G is not of types (1) or (2), but vert(G) = vert($X \cup Y \cup Z$) where X, Y, Z are directed cycles, $X \cap Y$ and $Y \cap Z$ are connected and non-empty, and $X \cap Z = \phi$. In this case, say G is a <u>three hump</u> digraph.

(4) G is versatile.

<u>Remark</u>: In fact, unless G falls under one of cases (1), (2) or (3), G "contains", in a sense to be made precise later, one of the versatile digraphs MIN.1, MIN.2 or MIN.3.

<u>Proof</u>: If G is versatile, then G falls under case (4) and we are finished. Thus suppose that G is not versatile. Since G is strongly connected, write vert(G) = $\cup_{i=1}^{m}$ vert(C_i) where the C_i are directed cycles of G, and for each j, $2 \leq j \leq m$,

there exists $i < j$ such that $C_j \cap C_i \neq \phi$.

Do this so that m is as small as possible.

If m = 1, then G is of type (1) and we are done. If m = 2 then G is of type (2), for by the intersection lemma, since G is not versatile, $C_1 \cap C_2$ must be connected.

If m = 3, $C_1 \cap C_2 \neq \phi$. Suppose without loss of generality that $C_2 \cap C_3 \neq \phi$. Otherwise $C_2 \cap C_3 = \phi$ so that $C_3 \cap C_1 \neq \phi$, and we interchange the roles of C_1 and C_2.

Because G is not versatile, by the intersection lemma, $C_1 \cap C_2$, $C_2 \cap C_3$ are connected. It remains to show that $C_3 \cap C_1 = \phi$. Suppose not.

Let $x_1 \in$ vert(C_1) \ vert($C_2 \cup C_3$). Such an x_1 exists, for otherwise we could write vert(G) = vert($C_2 \cup C_3$) where $C_2 \cap C_3$ is non-empty. This contradicts the minimality of m.

Similarly we can choose $x_2 \in$ vert(C_2) \ vert($C_1 \cup C_3$) and $x_3 \in$ vert(C_3) \ vert($C_2 \cup C_1$).

Now we use the second weaving lemma. Let S = { x_1, x_2, x_3 }, and $v = h^{\omega}(x_2)$ as before. The required path P(x_1, x_2) follows C_1 from x_1 to $C_1 \cap C_2$, then C_2 to x_2. We see that x_3 is not on P(x, y) because $x_3 \notin C_1 \cup C_2$. Similarly we can find P(x_2, x_1), P(x_2, x_3), P(x_3, x_2), P(x_3, x_1), P(x_1, x_3). We can walk v on G modulo the P(x_i, x_j), contradicting our assumption that G is not versatile. Here G is cognate, in some sense, to the triangle, MIN.1.

We conclude that if m = 3, then $C_1 \cap C_3 = \phi$, and G is a three hump digraph

If $m \geq 4$, we must get a contradiction. We will consider the cycles C_1, C_2, C_3, C_4. As in the previous case, we may assume $C_1 \cap C_2 \neq \phi$, $C_2 \cap C_3 \neq \phi$, and $C_3 \cap C_1 = \phi$.

<u>Case A:</u> $C_4 \cap C_2 \neq \phi$.

Then pick $x_1 \in \text{vert}(C_4) \setminus \text{vert} (C_1 \cup C_2 \cup C_3)$. We can do this by minimality of m. Pick $x_2 \in \text{vert} (C_3) \setminus \text{vert}(C_1 \cup C_2 \cup C_4)$. Such a x_2 exists, otherwise m could be reduced by discarding C_3. Again, pick $x_3 \in \text{vert}(C_1) \setminus \text{vert}(C_2 \cup C_3 \cup C_4)$

Again use the second weaving lemma with $S = \{ x_1, x_2, x_3 \}$ and $v = h^{\omega}(x_2)$. We can let $P(x_1, x_2)$ be a path from x_1 through C_4 to $C_4 \cap C_2$, through C_2 to $C_2 \cap C_3$, through C_3 to x_2. Clearly x_3 is not on this path. Similarly we choose $P(x_1, x_3)$, $P(x_2, x_1)$, $P(x_2, x_3)$, $P(x_3, x_1)$, $P(x_3, x_2)$.

By the second weaving lemma, G is versatile, which is a contradiction. (This case is cognate to the undirected graph case where G has a vertex v of degree 3 or greater. Here, C_2 plays the role of vertex v.)

<u>Case B:</u> $C_4 \cap C_2 = \phi$.

Suppose without loss of generality that $C_4 \cap C_3 \neq \phi$. Otherwise interchange the roles of C_1 and C_3. Now pick a vertex 1, with $1 \in \text{vert}(C_1) \setminus \text{vert}(C_2 \cup C_3 \cup C_4)$. Such a vertex exists because m is minimal. Pick vertex

2 $\in$ vert($C_1 \cap C_2$), vertex 3 $\in$ vert($C_2 \cap C_3$), vertex 4 $\in$ vert($C_3 \cap C_4$), and vertex 5 $\in$ vert(C_4) \ vert($C_1 \cup C_2 \cup C_3$). Let S = { 1, 2, 3, 4, 5 }, and walk w = g(h^{ω}(x_2)) on G modulo paths, where h, g are substitutions 2.1 and 2.2 from chapter 2.

The two letter subwords of v are 12, 23, 34, 45, 54, 43, 32, 21. Choose the paths P(1, 2), P(2, 1) in C_1. Since $C_1 \cap C_3 = \phi$, 3 and 4 are not on P(1, 2) or P(2, 1). Also 5 $\notin C_1$ so that 5 is not on P(1, 2) or P(2, 1). Let P(2, 3) and P(3, 2) be paths in C_2. These paths avoid 1 and 2 which are not on C_2, and 4 and 5 which are on C_4, as $C_2 \cap C_4 = \phi$. Choose P(3, 4), P(4, 3) in C_3 and P(4, 5), P(5, 4) in C_4. By arguments symmetrical to those used with the first four paths, these last four paths satisfy the conditions of the second weaving lemma. Thus G is versatile, which is a contradiction. The reader will perceive that we treat G as though it were a five element path. (MIN.3) □

The intersection and classification lemmas can be invoked to show that certain classes of digraphs are versatile. To show that an individual digraph is not versatile, it suffices to exhaust the non-repetitive walks on that particular digraph. Next, we provide ways to show that classes of digraphs do not allow arbitrarily long non-repetitive walks.

Lemma 3.4 (Compressible Paths Lemma): Let $a_1a_2...a_n$, $n \geq 2$, be a directed path in a digraph G with

outdegree(a_1) = 1,

degree(a_i) = 2, i = 2 to n-1,

indegree(a_n) = 1.

Then G is versatile if and only if G' is, where G' is obtained from G by removing a_2, a_3, ..., a_n, and adding an edge in G' from a_1 to every successor of a_n.
(i. e. We identify the vertices of the path.)

Proof: The result will follow by induction if we prove the lemma for n = 2. Suppose then, that n = 2.

Clearly if G' is versatile then G is, by the weaving lemma.

Suppose G is versatile. Let w be any non-repetitive walk in G with the sole restriction that w does not start with a_2 or end with a_1. Consider $w' = w_{|vert(G)\setminus\{ a_2 \}}$, the word obtained from w by deleting all occurrences of a_2. Clearly w' will be a walk on G'. If we can show that w' is non-repetitive, we shall be done, for
$| w' | \geq | w | / 2$, which can be made arbitrarily large.

If v is any word on vert(G) \ { a_2 }, then let p(v) be the word obtained from v by replacing each occurrence of a_1 in v by a_1a_2. Then clearly, p(w') = w.

Now suppose for the sake of contradiction that w' is repetitive, say that w' = abbc for some $a, b, c \in \text{vert}(G) \setminus \{ a_2 \}$, $b \neq \epsilon$. But then $p(w') = p(a)p(b)p(b)p(c)$, and w contains a repetition, which is a contradiction.□

Definition: Let G be a digraph so that all the vertices of G lie on a directed path P of G. Let ij be a directed edge of G not on P. If j precedes i in P, then the edge ij is a back edge (with respect to P). Otherwise, the edge ij is a forward edge (with respect to P).

Definition: Let G be a digraph with all its vertices on a directed path P, so that vert(G) is ordered. Let ij be a back edge of G. We say that edge ij is useful if one of the following cases arises:

(i) A forward edge kl of G has a vertex between j and i; $j \leq l \leq i$ or $j \leq k \leq i$ (or both).

(ii) There are two back edges of G, i'j' and i"j", such that $j \leq j' < j" \leq i' < i" \leq i$, but not both $j = j'$ and $i" = i$. We say that i'j' and i"j" form an M under ij.

(iii) A back edge kl of G intersects ij; that is, $l < j \leq k < i$ or $j < l \leq i < k$. We say that kl and ij form an M.

Otherwise say that ij is <u>useless</u>.

<u>Lemma 3.5 (M lemma):</u> Let G be a digraph with all its vertices on a directed path P. Let ij be a useless edge of G. Then G is versatile if and only if G \ ij is, where G \ ij is the graph obtained from G by removing the edge ij.

<u>Proof:</u> First note that removing an edge from G never makes another edge useful.

Next let Q be the set of back edges of G with partial order ⊃ : i"j" ⊃ i'j' if j" ≤ j' < i' ≤ i", viz. the ends of the smaller edge are between those of the larger.

It suffices to prove the lemma in the case that ij is minimal with respect to this order. Suppose that the lemma has been proved in this case and kl is any useless edge of G. Let the set of useless edges of G less than or equal to kl be S = { i_1j_1, i_2j_2, ..., i_nj_n, kl }. Then G \ S is versatile if and only if G is; we simply remove the edges of S from G one at a time, at each step removing a minimal edge. To get G \ kl, we add the edges of S \ { kl } to G \ S, starting with maximals.

Suppose then that ij is a useless edge of G, minimal in the order given. Let v b a non-repetitive word of type

w walkable on G. If ij appears only finitely often in v, then a final segment of v can be walked on G \ ij, and we are done. Thus assume that ij appears infinitely often as a subword in v. We can then find arbitrarily long subwords w of v such that w has i as a suffix.

<u>Claim</u>: Any long enough subword w of v having i as a suffix must have suffix j(j+1)(j+2)...(i-1)i. (Here j+1 is the successor of j on P etc.)

<u>Proof of Claim</u>: The indegree of i is 1: Any forward edge ending at i satisfies (i) of the definition of useful edges, making ij useful. Any back edge ending at i satisfies (iii) of the definition, making ij useful.

Thus w ends in (i-1)i.

Now suppose that long enough w ending in i must end in

(i-k)(i-k+1)...(i-1)i, $j < i-k$ .....(*

We show that w ends in (i-k-1)(i-k)...(i-1)i.

Suppose not. Then some edge e = l(i-k), $l \neq i-k-1$ exists in G, and w ends in l(i-k)...i. If $l < i-k$, then e is a forward edge satisfying (i) of the definition of useful edges, a contradiction.

Thus we must assume that e is a back edge. Because of (iii) of the definition of useful edges, we must have $l \leq i$. Since e is not a useless edge, by the minimality of ij, there are two possibilities:

I) There is an M under e. Such an M is also under ij, a contradiction, as per (ii) of the definition of useful edges.

II) Some edge f = rs forms an M with e where

$s < i-k \leq r < l$

or $i-k < s \leq l < r$.

Because of (ii), (iii) of the definition of useful edges, we insist that $j = s < i-k \leq r < l = i$

or $j = i-k < s \leq l < r = i$.

However by assumption, $j < i-k$, so we must have

$$j = s < i-k \leq r < l = i.$$

Thus w ends in l(i-k)...i = i(i-k)i. But then, if w is long enough, our induction hypothesis (* says that w ends in (i-k)...i(i-k)...i, and v contains a repetition, which is a contradiction.

Thus w ends in (i-k-1)(i-k)...i. By induction, w ends in j(j+1)...(i-1)i.□

A second claim has a similar proof.

<u>Claim</u>: Any long enough subword z of v having j as a prefix must have prefix j(j+1)(j+2)...(i-1)i.

However v contains ij infinitely often, so that we can find a long subword wijz of v with w = w'j(j+1)(j+2)...(i-1)i, z = j(j+1)(j+2)...(i-1)iz'. But then v contains the repetitive subword

$w'j(j+1)(j+2)\ldots(i-1)ij(j+1)(j+2)\ldots(i-1)iz'$, a contradiction. We conclude that v contains ij only finitely often, and thus $G \setminus ij$ is versatile if and only if G is.□

Clearly the existence of a loop in a digraph does not help to make it versatile. We may therefore modify Lemma 3.4 slightly:

<u>Lemma 3.6 (Compressible Paths Lemma):</u> Let $a_1a_2\ldots a_n$ be a directed path in a digraph G with

outdegree(a_1) = 1,

degree(a_i) = 2, i = 2 to n-1,

indegree(a_n) = 1.

Then G is versatile if and only if G' is, where G' is obtained from G by removing a_2, a_3, ..., a_n, and adding an edge in G' from a_1 to every successor of a_n other than a_1.

We say that digraph G <u>reduces</u> to digraph H (H is a <u>reduction</u> of G) if H is obtained from G by repeated applications of the compressible paths lemma and removal of loops and useless edges. Thus if G reduces to H, G is versatile if and only if H is versatile.

The purpose of this thesis is to characterize versatile digraphs. We make this characterization by producing two sets of digraphs, MIN (shown in Appendix 1) and MAX (shown in Appendix 2). In Chapters 7 and 8 we show that the digraphs of MIN are versatile. In Chapter 9, we show that the digraphs of MAX are not versatile. In Chapter 4, Chapter 5 and Chapter 6, the heart of the thesis, we give a case by case breakdown of all digraphs to show that every digraph either can be reduced to some digraph "contained" in a digraph of MAX, and hence is non-versatile, or else "contains" some digraph of MIN, and hence is versatile. The intersection lemma, the classification lemma, the M lemma, and the definitions of useless edges, forward edges and back edges will be used to give this case breakdown of digraphs. The next section of this chapter introduces the concept of <u>mimicking</u>, by which we make precise what it means for a digraph G to "contain" a digraph H .

<u>Definition:</u> Let H, G be digraphs so that there is an injection m: vert H -> vert G, such that whenever ij is an edge of H, then there is a path in G \ m(vert H) from m(i) to m(j). We say that G <u>imitates</u> H.

We can put this another way: We fix a labelling of G. Whenever v is a walk on H then v can be walked on G modulo paths with respect to this labelling. It follows that if G imitates H, then if H is versatile, so is G.

Example: The graph of Figure 3.1 imitates the triangle with the given labelling.

Not every versatile digraph imitates P_5 or the triangle. (Otherwise we would be finished, by Chapter 2.) The digraph G of Figure 3.2 is a counterexample. This graph is indeed versatile, because the following substitution is suitable.

g: x_1 -> 1232

x_2 -> 123454

x_3 -> 123456

This is easy to check, or refer to the Different Endings Lemma of Chapter 7. However, an argument could be given to show that G can imitate neither the triangle nor the five element path.

If G is a digraph, then G^R, the reverse of G, is the digraph with the same vertex set as G, and a directed edge ij exactly when ji is a directed edge of G. Clearly G^R is versatile if and only if G is. To reduce the size of MIN, we have sought to include at most one of G and G^R for any digraph G. Let us extend the idea of imitation to take advantage of this:

Definition: Let H, G be digraphs. Say that G mimics

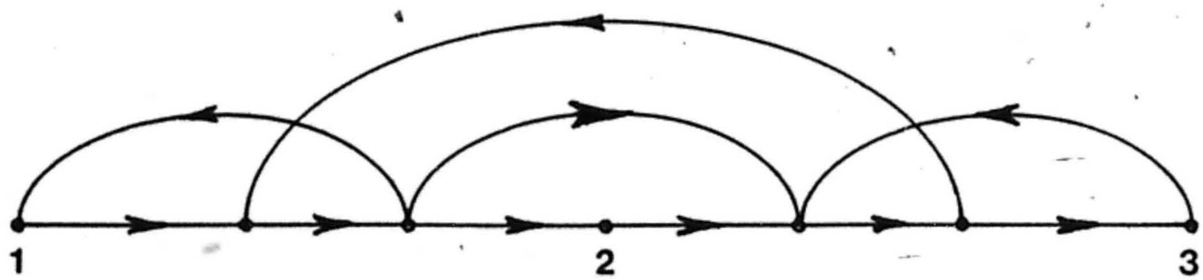


Figure 3.1

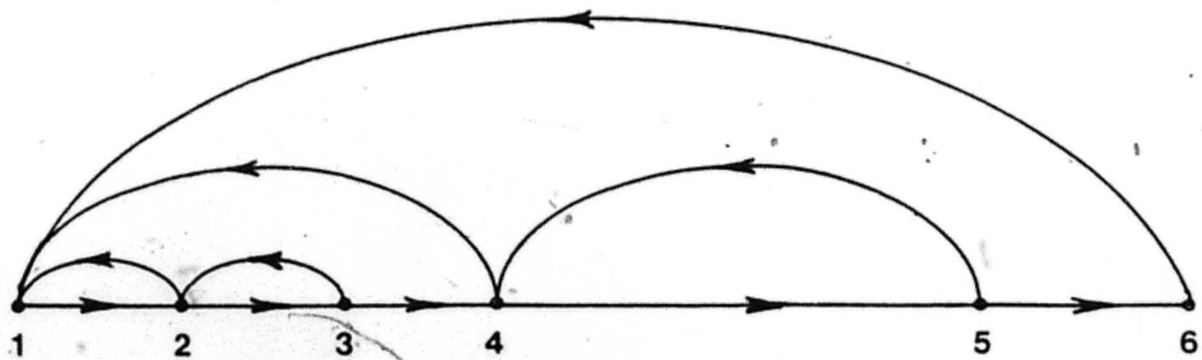


Figure 3.2

H if G imitates at least one of H, H^R.

Now that we have introduced the concept of mimicking, we remark that the proofs of Lemmas 3.2 and 3.3 prove the following stronger results:

Lemma 3.2' (Intersection Lemma): Let X, Y be directed cycles in the digraph G so that vert(X) $\cap$ vert(Y) $\neq \phi$. Then either

1) X $\cap$ Y is connected

or 2) X $\cup$ Y mimics one of MIN.1 or MIN.2, and hence is versatile

Lemma 3.3' (Classification Lemma): Let G be a strongly connected digraph . Then G is of one of the following types:

(1) vert(G) = vert(X) for some directed cycle X of G. In this case, say G is a one hump digraph.

(2) G is not of type (1), but vert(G) = vert (X $\cup$ Y) where X and Y are directed cycles, and X $\cap$ Y is connected and non-empty. In this case, say G is a two hump digraph.

(3) G is not of types (1) or (2), but vert(G) = vert(X $\cup$ Y $\cup$ Z) where X, Y, Z are directed cycles, X $\cap$ Y and Y $\cap$ Z are connected and non-empty, and

$X \cap Z = \phi$. In this case, say G is a three hump digraph.

(4) G mimics one of MIN.1, MIN.2 or MIN.3, and therefore is versatile.

We now have the tools necessary to state and prove our main result. The main theorem of this work is proved in three pieces, appearing in Chapter 4, Chapter 5 and Chapter 6, respectively.

Theorem 3.8: Let G be a three hump digraph. Either G mimics a graph H where H is in MIN, or a reduction of G is mimicked by some digraph K, where K is in MAX.

Theorem 3.9: Let G be a two hump digraph. Either G mimics a graph H where H is in MIN, or a reduction of G is mimicked by some digraph K, where K is in MAX.

Theorem 3.10: Let G be a one hump digraph. Either G mimics a graph H where H is in MIN, or a reduction of G is mimicked by some digraph K, where K is in MAX.

The Main Theorem (Theorem 3.11): Let G be a digraph. Either G mimics a graph H where H is in MIN, or a reduction of G is mimicked by some digraph K, where K is in MAX.

Corollary 3.12: Let G be any digraph. Either G is non-versatile, or else G mimics a graph H in MIN.

Chapter 4: Three Hump Digraphs

In this chapter we prove Theorem 3.8.

Theorem 3.8: Let G be a three hump digraph. Either G mimics a graph H where H is in MIN, or a reduction of G is mimicked by some digraph K, where K is in MAX.

We begin by proving a refinement of the classification lemma.

Lemma 4.1 (Refining the Classification Lemma): Let G be a three hump digraph . Then either

(1) vert(G) = vert ($X \cup Y \cup Z$) where X, Y, Z are cycles of G,

$X \cap Y$, $Y \cap Z$ are connected and non-empty, $X \cap Z = \phi$, and

$Y \setminus (X \cup Z)$ is connected.

or (2) G is versatile. In fact G mimics MIN.4.

Proof: Suppose that $Y \setminus (X \cup Z)$ is not connected. Then choose vertices $1 \in X \setminus Y$, $2 \in X \cap Y$, $4 \in Y \cap Z$ and $5 \in Z \setminus Y$. Pick two vertices 3, 3' from different components of $Y \setminus (X \cup Z)$. Without loss of generality we may assume that vertices 2, 3, 3', 4 appear in cyclical order 2, 3, 4, 3' in Y. (Recall that $X \cap Y$, $Z \cap Y$ are connected.) With this labelling, G mimics MIN.4. (See Figure 4.1.)□

This refinement of the classification lemma allows us to

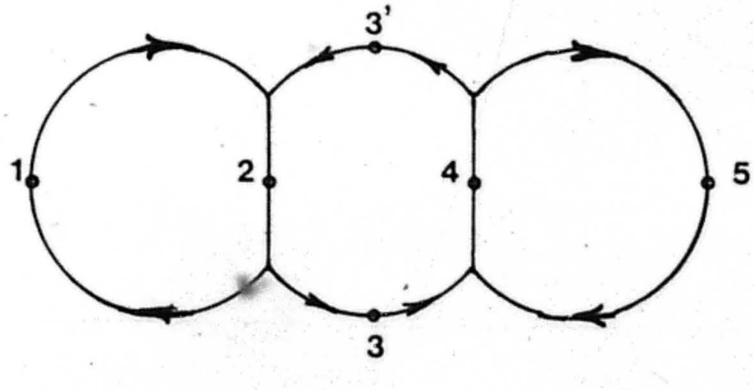


Figure 4.1

introduce a certain structure to three hump digraphs.

Definition: Let G be a three hump digraph. We say that G has a skeleton if

(1) We can write vert(G) = vert(P) where P is a directed Hamiltonian path in G. Path P gives an order to the vertices of G.

(2) With respect to this order, G has at least three additional edges a_2a_1, b_2b_1, c_2c_1 where $a_1 < b_1 \leq a_2 < c_1 \leq b_2 < c_2$, a_1 is the initial vertex of P, c_2 the final vertex.

We call the digraph made up of P together with the edges a_2a_1, b_2b_1, c_2c_1 the skeleton of G. (See Fig. 4.2.) Other edges of G are called extra-skeletal edges.

Lemma 4.2 (The Skeleton): Let G be a three hump digraph which does not mimic MIN.4. Then G has a skeleton.

Proof: We may assume by Lemma 4.1 that Y \ (X ∪ Z) is a directed path. Let m be the source of this directed path and M the sink. Let a_2 be the predecessor of m in Y. Either $a_2 \in X$ or $a_2 \in Z$, but not both. Suppose without loss of generality (up to renaming) that $a_2 \in X$. Let a_1 be the successor of a_2 in X.

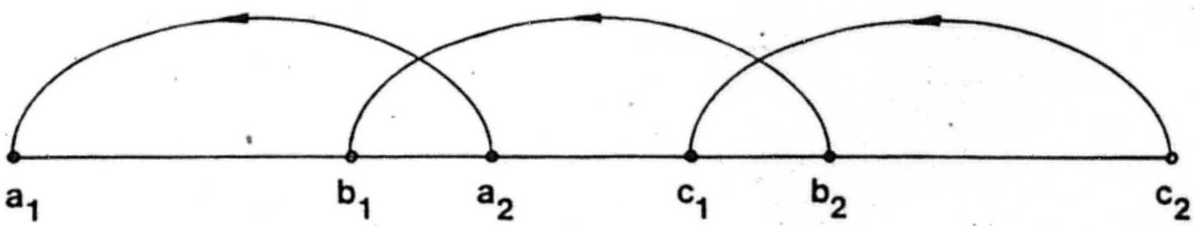


Figure 4.2

Let c_1 be the successor of M in Y. Then $c_1 \in Z$. Otherwise $c_1 \in X$, and Y is the union of two directed paths: the segment of Y from m through M, and the segment of X from c_1 through a_2. (Recall that $Y \cap X$ is connected.) But then $Y \cap Z = \phi$, since $X \cap Z = \phi$ and $(Y \setminus (X \cup Z)) \cap Z = \phi$. This is a contradiction. Thus indeed $c_1 \in Z$. Let the predecessor of c_1 in Z be c_2.

Now $X \cap Y$ is a directed path with a_2 as sink. Let b_1 be the source. Let b_2 be the sink of the directed path $Y \cap Z$, which has c_1 as source. (See Figure 4.3.)

Now X is the cycle $a_1 — a_2$, $Y \setminus (X \cup Z)$ is the path $m — M$, and Z is the cycle $c_1 — c_2$. We may therefore let $P = a_1 — a_2 m — M c_1 — c_2$. Clearly we have vert(G) = vert(P),

$a_1 \leq b_1 \leq a_2 \leq c_1 \leq b_2 \leq c_2$,

a_1 is the initial vertex of P, c_2 the final vertex of P.

We show that $a_1 < b_1$, $a_2 < c_1$, $b_2 < c_2$.

If $a_1 = b_1$, then $X = a_1 — a_2 = b_1 — a_2 \subset X \cap Y$, so that $X \subset Y$. Then vert(G) = vert ($Y \cup Z$), a contradiction.

Similarly $b_2 \neq c_2$.

Finally, $a_2 \neq c_1$, as $X \cap Z = \phi$.

The edges a_2a_1, c_2c_1 exist by definition. The edge

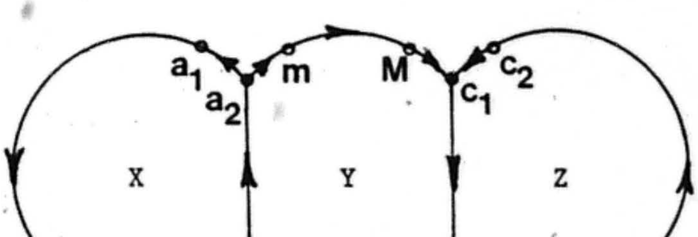


Figure 4.3

b_2b_1 exists because Y is a cycle.□

Proof of Theorem 3.8: We assume that G has a skeleton, since otherwise, by Lemma 4.2 above, G mimics MIN.4. Also, we may assume that G has no useless edges, as such edges may be removed without affecting whether G is versatile or not. The proof of the theorem involves a lengthy enumeration of cases. To make this case breakdown we refer to the skeleton of G (Figure 4.2). Let the extra-skeletal edges of G be i_1j_1, i_2j_2,..., i_mj_m. We make cases based on m.

To reduce work, we often invoke symmetry. Now G^R, the reverse of G is a three hump digraph. Again, vert(G^R) = vert(P^R) where P^R is the reverse of P. Renaming a_1 as c_2', a_2 as c_1', b_1 as b_2', b_2 as b_1', c_1 as a_2' and c_2 as a_1', we see that G^R is a three hump digraph with skeleton $P^R \cup \{ c_2'c_1', b_2'b_1', a_2'a_1' \}$. This symmetry under reversal reduces the number of required cases. For example, suppose G has an edge i_1j_1 with $i_1 > b_2$. Then, renaming i_1 as j_1' and j_1 as i_1', the reversal of G has $j_1' < b_1'$ and will later fall under our case A1. Keeping this use of symmetry in mind, we procede to our case division.

m = 0: If G is its own skeleton then a reduction of G can be mimicked on MAX.1 and we are done. (See Figure

4.2.) We apply the Compressible Paths Lemma, Lemma 3.6, to the paths in G

from a_1 to the predecessor in P of b_1

from b_1 to a_2

from the successor of a_2 to the predecessor of c_1

from c_1 to b_2

and from b_2 to c_2.

The result is isomorphic to a graph in one of Figure 4.4 or Figure 4.5, depending on whether there is a vertex between a_2 and c_1 in G. These graphs are mimicked by MAX.1 with the given labellings.

<u>m = 1:</u> Depending on i_1, j_1 we have several subcases.

<u>Case A:</u> The edge $i_1 j_1$ is a back edge; i.e. With respect to the order given to vert(G) by P, $i_1 > j_1$.

<u>Case B:</u> The edge $i_1 j_1$ is a forward edge; i.e. $i_1 < j_1$.

Case A breaks down as follows:

Case A1 $a_1 \leq j_1 < b_1$ (or symmetrically, $i_1 > b_2$).

Case A2 $b_1 \leq j_1 \leq a_2$ (and $i_1 \leq b_2$).

Case A3 $a_2 < j_1 < c_1$ (and $i_1 < c_1$).

In this third case, the edge $i_1 j_1$ is useless, a contradiction. (See Figure 4.6)

Figure 4.4

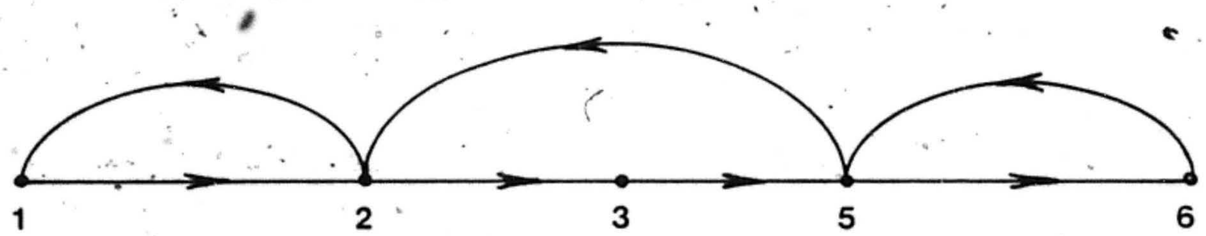


Figure 4.5

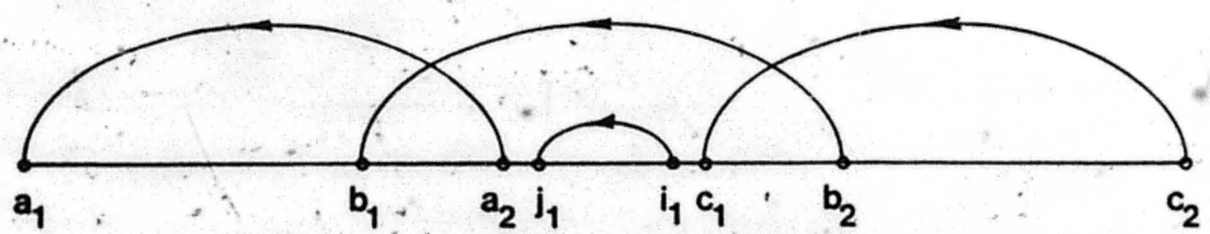


Figure 4.6

Cases A1 and A2 are further subdivided.

A1: Case A1 (a) $a_1 \leq j_1 < i_1 < b_1$.

Here the edge $i_1 j_1$ is useless, a contradiction. (See Figure 4.7.)

Case A1 (b) $a_1 \leq j_1 < b_1 \leq i_1 < a_2$.

Here G mimics MIN.5. (See Figure 4.8.) The labelling of vertices of G required by the definition of mimicking is shown explicitly in the figure.

Case A1 (c) $a_1 \leq j_1 < b_1 \leq a_2 \leq i_1 < c_1$.

If $a_2 = i_1$, then $a_1 < j_1$ so that G mimics MIN.6. (See Figure 4.9.)

If $a_2 < i_1$, then G mimics MIN.5. (See Figure 4.10.)

Case A1 (d) $a_1 \leq j_1 < b_1 < c_1 \leq i_1$

Note that $i_1 < c_2$ and $a_1 < j_1$ or vert(G) could be written as the union of two cycles.

Here G mimics MIN.7. (See Figure 4.11.)

A2: Case A2 (a) $b_1 \leq j_1 < i_1 \leq a_2$. Here the edge $i_1 j_1$ is useless, a contradiction. (See Figure 4.12.)

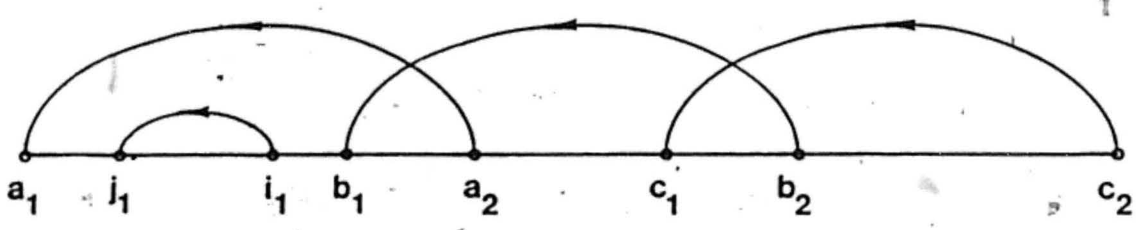


Figure 4.7

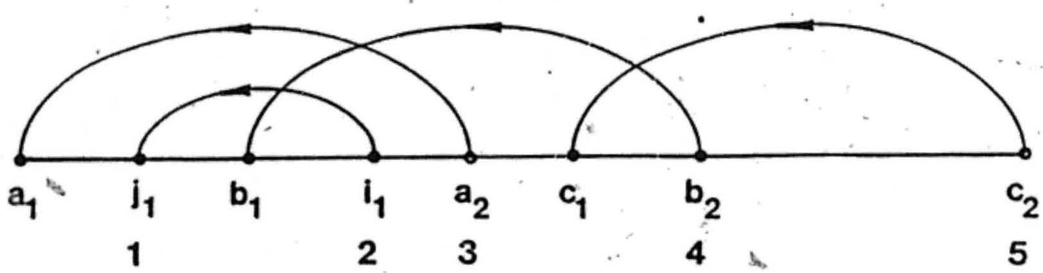


Figure 4.8

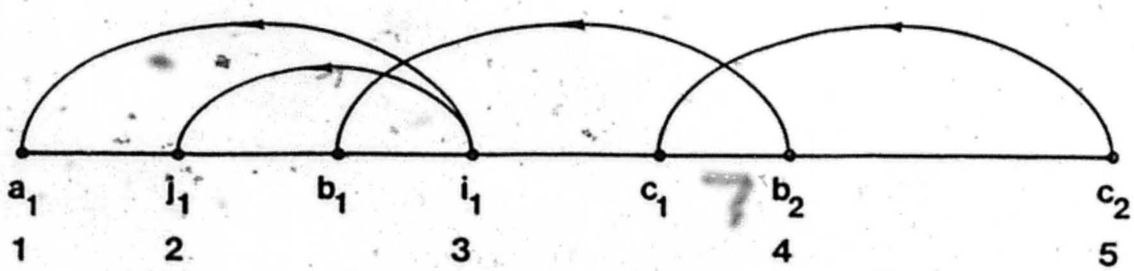


Figure 4.9

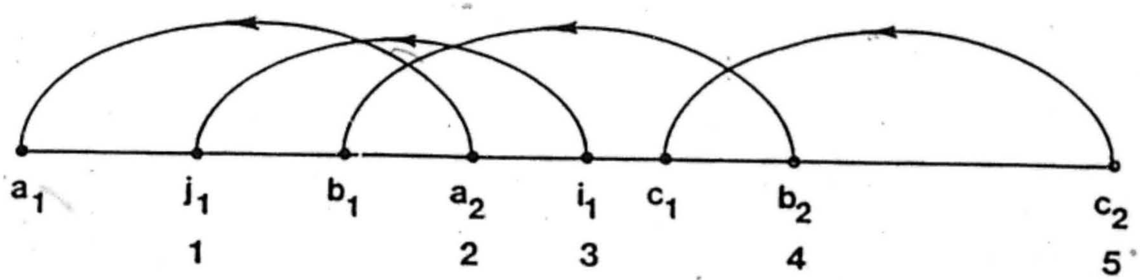


Figure 4.10

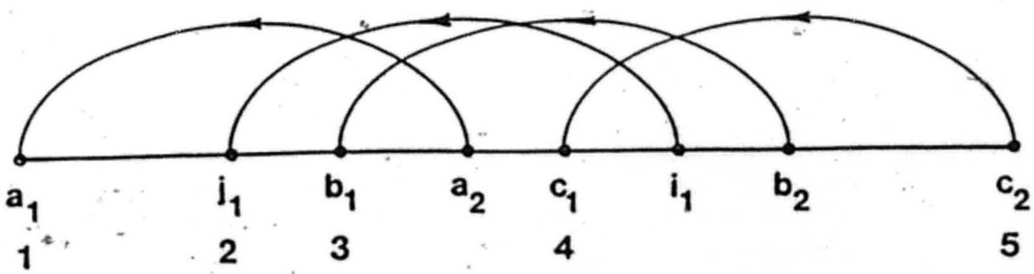


Figure 4.11

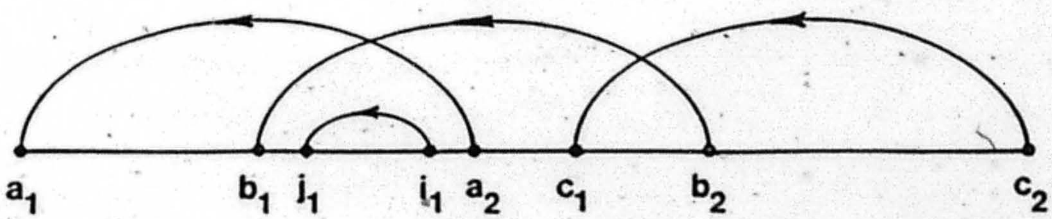


Figure 4.12

Case A2 (b) $a_2 < i_1 < c_1$.

If $j_1 = b_1$ then the reduction of G can be mimicked on MAX.1 (See Figure 4.13.) and we are done.

If $j_1 > b_1$ then G mimics MIN.8. (See Figure 4.14.)

Case A2 (c) $c_1 \le i_1 \le b_2$.

Either $j_1 \ne b_1$ or $i_1 \ne b_2$, since $i_1j_1 \ne b_2b_1$. Since $b_1 \le j_1 \le a_2$ and $c_1 \le i_1 \le b_2$, the roles of i_1 and j_1 are reversed when G is reversed. (See Figure 4.15.) Therefore, without loss of generality, suppose that $j_1 \ne b_1$. This takes us from Figure 4.15 to Figure 4.16.

But now $a_1 < j_1 \le a_2 < c_1 \le i_1 < c_2$, and $a_1 < b_1 < j_1$. Thus j_1 and i_1 can play the roles of b_1 and b_2 in the skeleton of G. Switching the roles of i_1j_1 and b_2b_1 gives case A1 (d) which has already been dealt with.

This concludes case A.

Case B is divided as follows:

Case B1: $i_1 < b_1$

Case B2: $b_1 \le i_1 \le a_2$ ($j_1 \le b_2$).

Case B3: $a_2 < i_1 < j_1 < c_1$.

In Case B3, let x be a vertex between i_1 and j_1 on

Figure 4.13

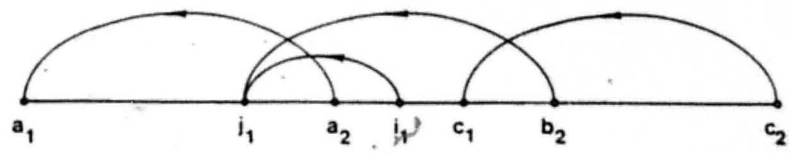


Figure 4.14

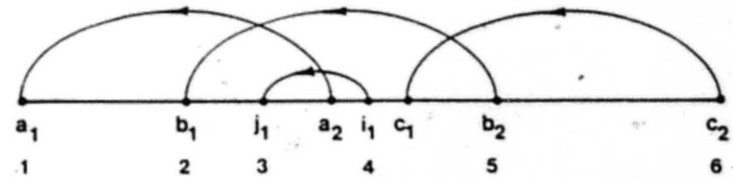


Figure 4.15

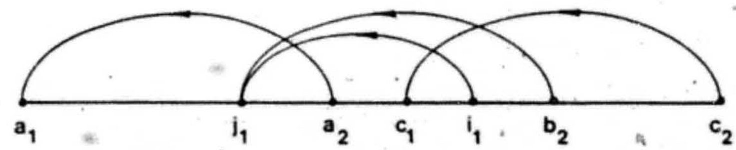


Figure 4.16

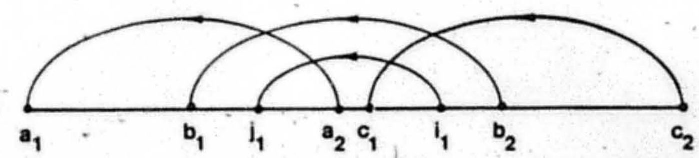
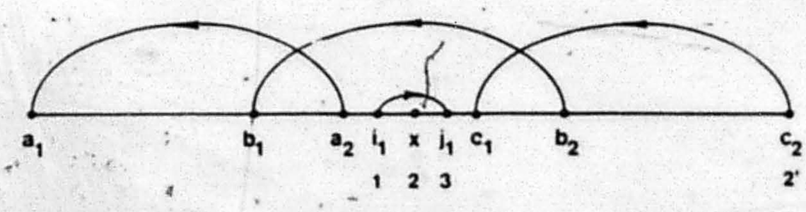


Figure 4.17



P. Then G mimics MIN.2. (See Figure 4.17.)

Cases B1 and B2 are further subdivided.

B1: Case B1 (a): $j_1 < b_1$

Let x be a vertex between i_1 and j_1 on P. Here G mimics MIN.2. (See Figure 4.18.)

Case B1 (b): $j_1 = b_1$.

Here G mimics MIN.9. (See Figure 4.19.)

Case B1(c): $b_1 < j_1 \leq a_2$.

Here G mimics MIN.10. (See Figure 4.20.)

Case B1 (d): $a_2 < j_1 \leq b_2$.

Consider the cycle C following P from a_1 to i_1, edge $i_1 j_1$, P from j_1 to b_2, edge $b_2 b_1$, P from b_1 to a_2, then edge $a_2 a_1$. Recall the cycle Z from the proof of Lemma 4.2: Z is the cycle consisting of the path in P from c_1 to c_2, together with the edge $c_2 c_1$. We see that $C \cap Z$ is connected. Therefore, vert(G) $\neq$ vert($C \cup Z$), as G is a three hump digraph. We have two possibilities:

(i) There is a vertex x between i_1 and b_1

Here G mimics MIN.11. (See Figure 4.21.)

Figure 4.18

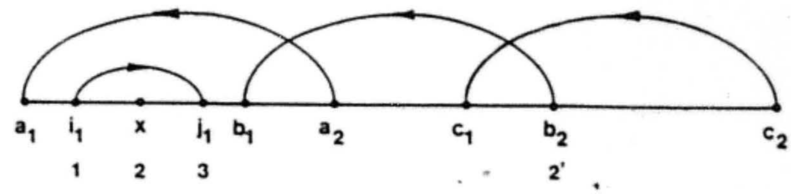


Figure 4.19

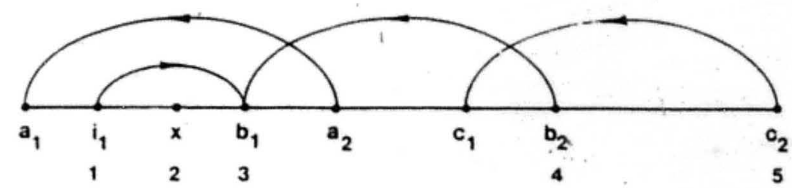


Figure 4.20

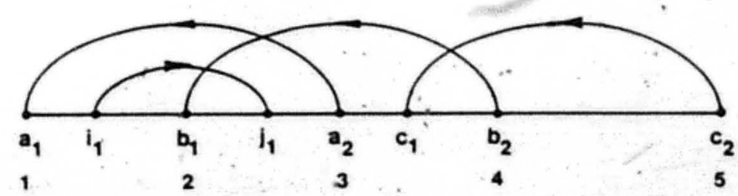
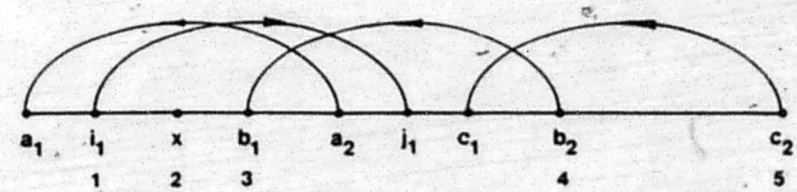


Figure 4.21



(ii) There is a vertex x between a_2 and the lesser of j_1, c_1.
Here G mimics MIN.12. (See Figure 4.22.)

Case B1 (e): $b_2 < j_1$

Consider the cycle C following P from a_1 to i_1, then edge i_1j_1, then P from j_1 to c_2, then edge c_2c_1, then P from c_1 to b_2, then edge b_2b_1, then P from b_1 to a_2, then edge a_2a_1. Now vert(G) cannot equal vert(C ∪ X), vert(C ∪ Y) or vert(C ∪ Z). This forces one of two cases:

(i) There is a vertex x of P between a_2 and c_1.
Here G mimics MIN.1. (See Figure 4.23.)

(ii) There are vertices of P between i_1 and b_1 and between b_2 and j_1.
Here G mimics MIN.13. (See Figure 4.24.)

B2: Case B2 (a) $j_1 < c_1$.

Here G mimics MIN.2. (See Figure 4.25. Let x be any vertex between i_1 and j_1.)

Case B2 (b) $c_1 \leq j_1 \leq b_2$.

We make two cases:

(i) There is some vertex x of G, $a_2 < x < c_1$. Here G

Figure 4.22

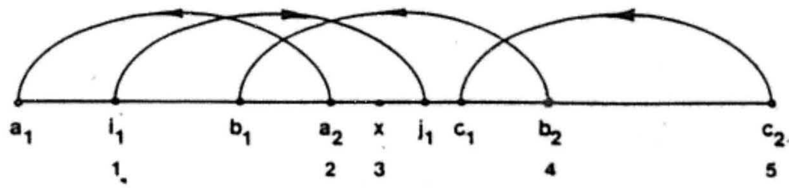
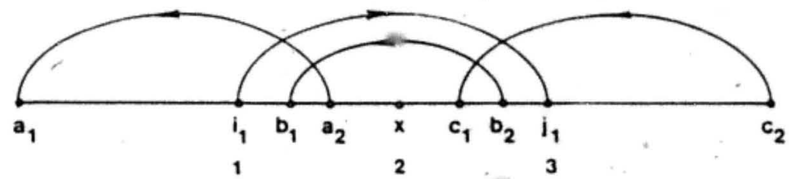


Figure 4.23



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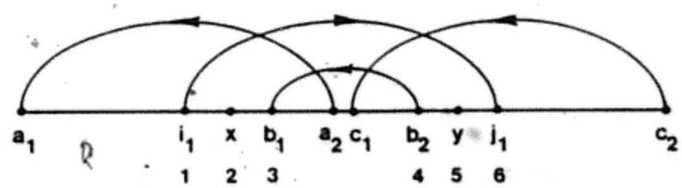

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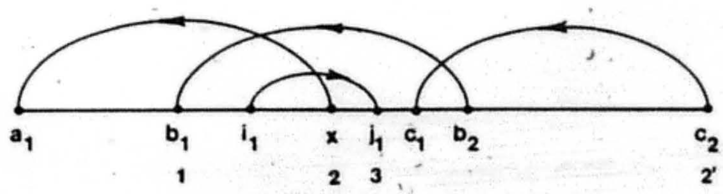
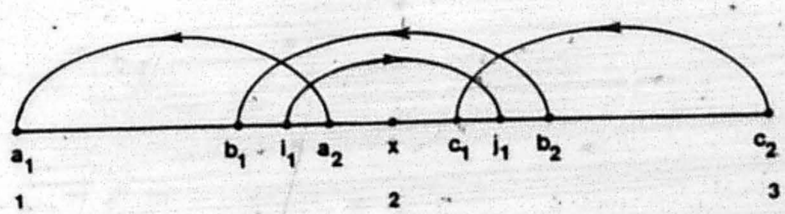


Figure 4.26



mimics MIN.1. (See Figure 4.26.)

(ii) There is no vertex of G between a_2 and c_1 on P. Then we cannot have $i_1 = a_2$ and $j_1 = c_1$, as i_1j_1 was chosen to be an extra-skeletal edge. By the symmetry of this case under reversal, we may assume that $i_1 \neq a_2$. Replace P by the hamiltonian path H. H starts with the successor of i_1, follows P to a_2, then follows edge a_2a_1 to get to a_1. Then H follows P from a_1 to i_1, then i_1j_1 to j_1. Next, H follows P from j_1 to c_2. If $c_1 = j_1$, then H stops at c_2. Otherwise, H follows edge c_2c_1 to c_1, then P to the predecessor of j_1. (See Figure 4.27.) With respect to the new skeleton, G falls under case B1(e), which has already been dealt with.

This completes the case when m = 1.

<u>m > 1:</u>

Without loss of generality we can assume that edge i_1j_1 falls (up to reversal of G) under one of cases A1(a), A2(a), A2(b) or A3 of the classification for m = 1. This is true because we have shown that if G contains an edge i_1j_1 falling under one of the other cases, G mimics a graph of MIN. Likewise assume that every other extra-skeletal edge of G falls under one of these cases.

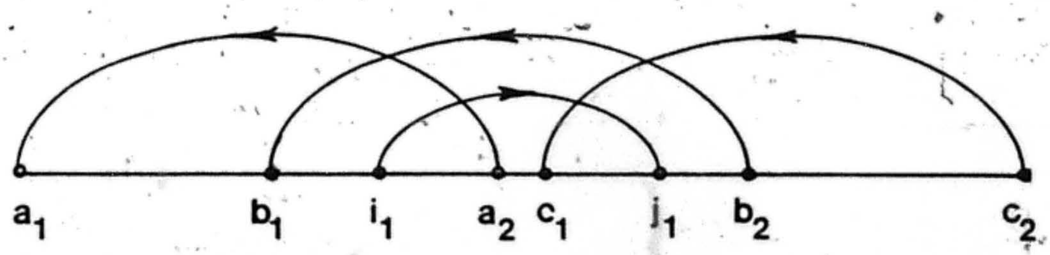


Figure 4.27

(Under the appropriate renaming, of course.) We thus use these cases for the breakdown of the present case.

Case A1 (a) $a_1 \le j_1 < i_1 < b_1$.

Suppose that edge $i_1 j_1$ falls under case A1 (a). Since $i_1 j_1$ is not a useless edge, and G has no forward edges, either there is an M under edge $i_1 j_1$, or an edge forms an M with edge $i_1 j_1$. However, any edge under $i_1 j_1$ is an edge of case A1 (a). Likewise, of the four types of edges remaining, only those falling under case A1 (a) could form an M with $i_1 j_1$. Thus without loss of generality (up to renaming), say that edges $i_1 j_1$ and $i_2 j_2$ form an M, with $a_1 \le j_1 < j_2 \le i_1 < i_2 < b_1$. Here G mimics MIN.14. (See Figure 4.28.)

Case A2 (a) $b_1 \le j_1 < i_1 < a_2$.

Without loss of generality (up to renaming), edges $i_1 j_1$ and $i_2 j_2$ form an M, $b_1 \le j_1 < j_2 \le i_1 < i_2 \le a_2$. Here G mimics MIN.15. (See Figure 4.29.)

Case A3 $a_2 \le j_1 < i_1 < c_1$.

Without loss of generality (up to renaming), edges $i_1 j_1$ and $i_2 j_2$ form an M. However, we now have two possibilities:

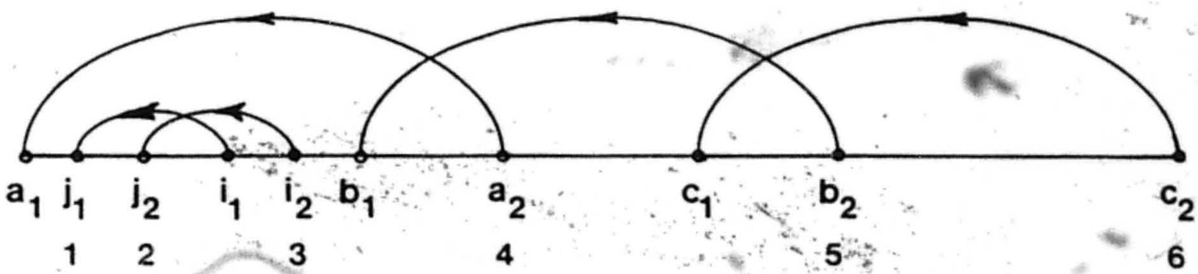


Figure 4.28

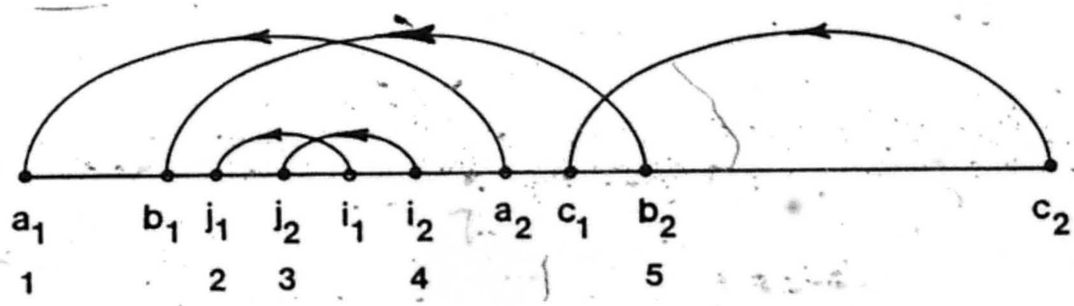


Figure 4.29

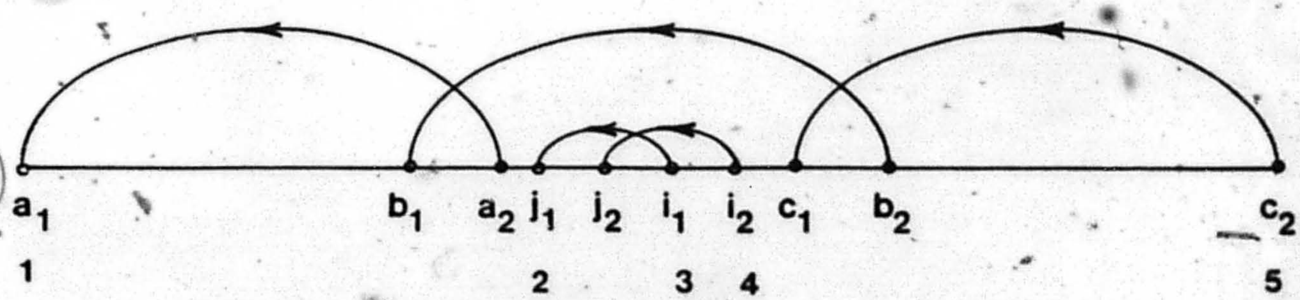


Figure 4.30

(i) Edge i_2j_2 falls under case A3 (after appropriate renaming.) Without loss of generality (up to renaming),

$a_2 < j_1 < j_2 \le i_1 < i_2 < c_1$.

Here G mimics MIN.15. (See Figure 4.30.)

(ii) Edge i_2j_2 falls under case A2 (b) (after appropriate renaming.) Without loss of generality (up to renaming),

$b_1 = j_2 \le a_2 < j_1 \le i_2 < i_1 < c_1$.

Here G mimics MIN.16. (See Figure 4.31.)

Case A2 (b) $b_1 = j_1 \le a_2 < i_1 < c_1$.

Without loss of generality, we may now assume that every extra-skeletal edge of G falls under case A2(b). However with reversals, this allows three possibilities.:

(i) We have $b_1 = j_2$, $i_1 < i_2 < c_1$. Here G mimics MIN.5.

(See Figure 4.32.).

(ii) We have $b_2 = i_2$, $j_2 \le i_1$. Here G mimics MIN.3. (See Figure 4.33.)

(iii) We have $b_2 = i_2$, $j_2 > i_1$. Here G mimics

Figure 4.31

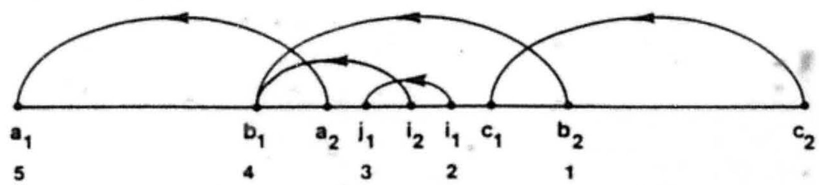


Figure 4.32

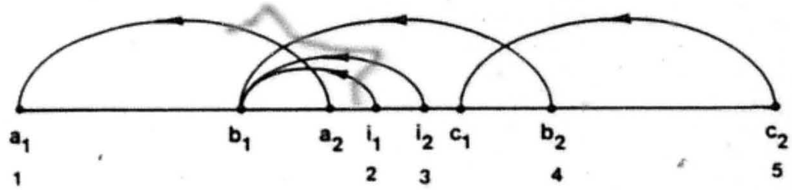


Figure 4.33

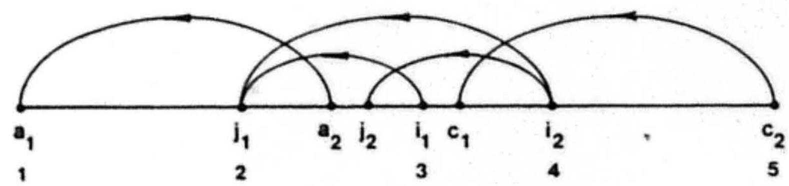
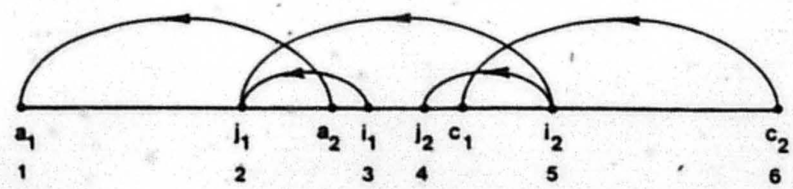


Figure 4.34



MIN.17. (See Figure 4.34.)

We have now proved the theorem.

Chapter 5: Two Hump Digraphs

In this chapter we will consider two hump digraphs. We prove

Theorem 3.9: Let G be a two hump digraph. Then either G mimics a digraph H in MIN, or a reduction of G is mimicked by a digraph K in MAX.

In analogy to the previous chapter, we introduce skeletons.

Definition: Let G be a two hump digraph. Then we say that G **has a skeleton** if

(1) We can write vert(G) = vert(P) where P is a directed Hamiltonian path in G.

(2) G has at least two additional edges a_2a_1, b_2b_1 where

$a_1 < b_1 \leq a_2 < b_2$ with respect to the order P induces on vert(G), a_1 is the initial vertex of P, b_2 the terminal vertex of P.

We call the digraph made up of P together with the edges a_2a_1, b_2b_1 the **skeleton** of G . (See Figure 5.1) Other edges of G are called **extra-skeletal** edges.

Lemma 5.1 (The Skeleton): Let G be a two hump digraph. Then G has a skeleton.

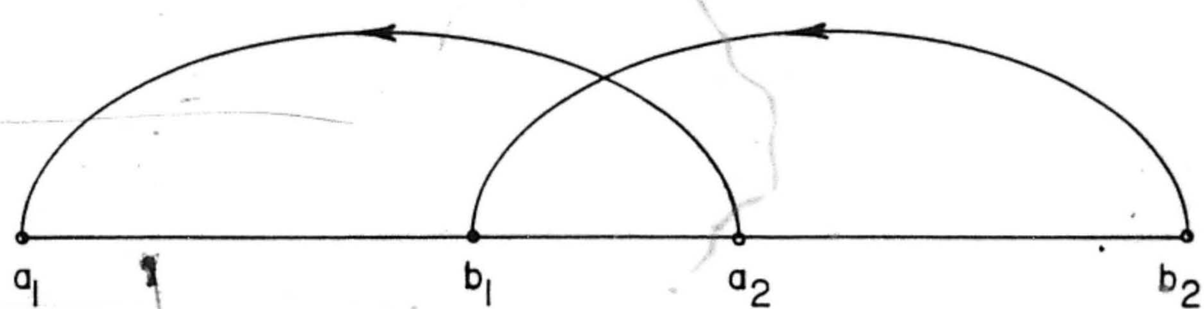


Figure 5.1

Proof: We know that $Y \cap X$ is a directed path. Let b_1 be the source of this directed path and a_2 the sink. Let b_2 be the predecessor of b_1 in Y. Let a_1 be the successor of a_2 in X. (See Figure 5.2)

Then the vertices of G all lie on the directed path a_1 — b_1 — a_2 — b_2. The edges a_2a_1, b_2b_1 exist by definition. Finally, in the case that $a_1 = b_1$ or $a_2 = b_2$, vert(G) lies on a cycle, X or Y respectively.□

Remark: The roles of X and Y in the previous proof are interchangeable.

Proof of Theorem 3.9: This proof involves a very long enumeration of cases, classifying the two hump digraphs. Assume again that G has no useless edges. Again the case breakdown refers to the skeleton of G. Label the extra-skeletal edges of G by i_1j_1, i_2j_2, ..., i_mj_m. We make cases based on m.

$m = 0$: If G is its own skeleton we are done. Here a reduction of G can be mimicked by MAX.1. (See Figure 5.1)

$m = 1$: We have two branches to our case division:

Case I: The edge i_1j_1 is a back edge; i.e. $i_1 > j_1$ with respect to the order given by P.

Case II: The edge i_1j_1 is a forward edge; i.e.

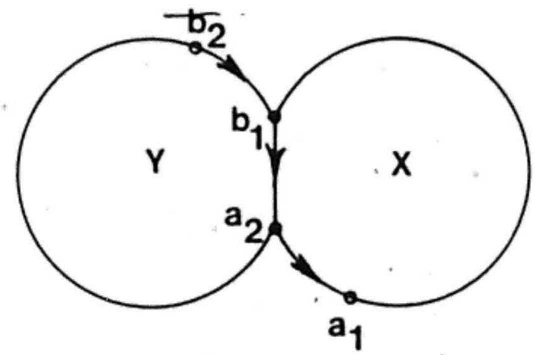


Figure 5.2

$i_1 < j_1$.

CASE I (ONE BACK EDGE)

Case I gives rise to several subcases. If $a_1 \leq j_1 < b_1$, then we form cases based on i_1.

Case A $a_1 \leq j_1 < b_1$ and $i_1 < b_1$.

Here the edge $i_1 j_1$ is useless, a contradiction. (See Figure 5.3)

<u>Note:</u> Later on in the proof, when we consider the possibility that $m > 1$, it will be useful to have names for the various types of back edges occurring in G. When $m = 1$, we have 5 subcases of case I, viz. cases A, B, C, D and E. We call an edge $i_r j_r$ of G a <u>type A</u> (B, C, D, E) edge if the graph G', formed by removing from G all extra-skeletal edges other than $i_r j_r$, falls under case A (B, C, D, E) of the present discussion.

Case B $a_1 = j_1$ and $b_1 \leq i_1 \leq a_2$.

(Therefore $i_1 \neq a_2$.)

A reduction of G can be mimicked by MAX.12. (See Figure 5.4)

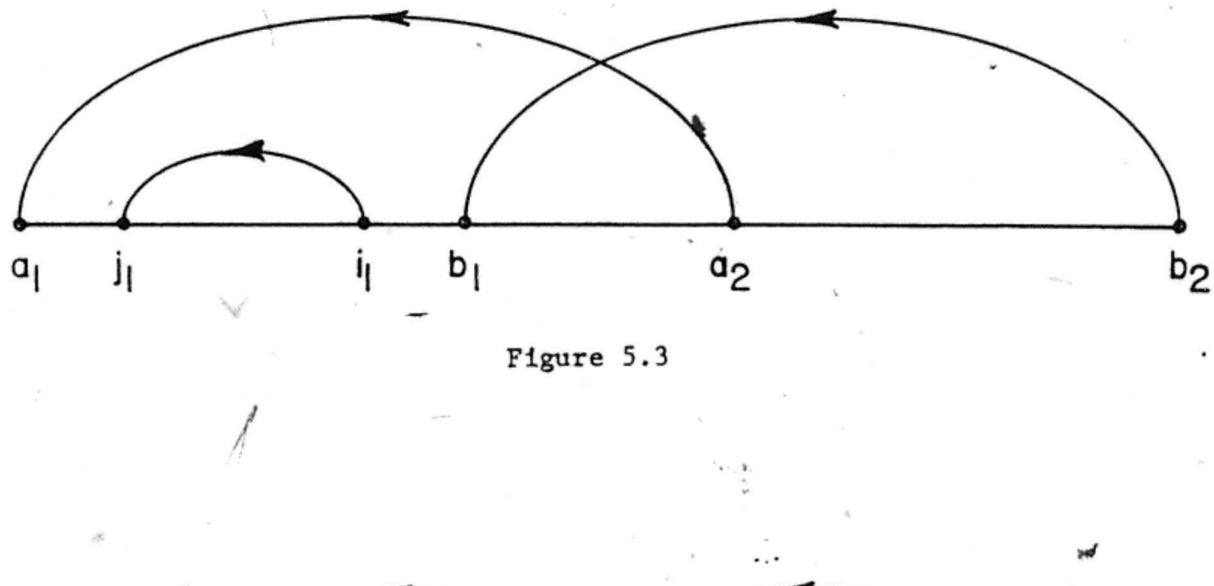


Figure 5.3

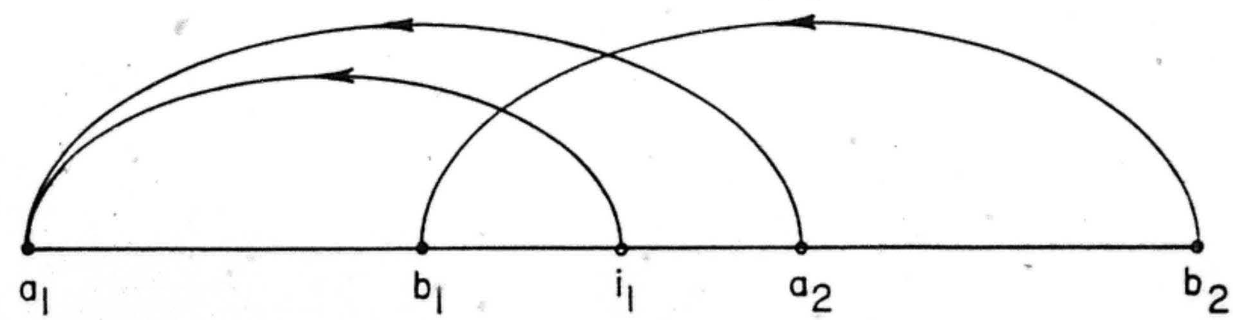


Figure 5.4

Case C $a_1 < j_1 < b_1$ and $b_1 \leq i_1 \leq a_2$. A reduction of G can be mimicked by MAX.2. (See Figure 5.5)

In the next case we will want to invoke symmetry. In preparation, we note that a digraph G is versatile if and only if G^R, the reverse of G, is versatile. It is useful now to extend our concept of type A, B, C, D, E edges to reverse edges. An edge $i_r j_r$ is a reverse type A (B, C, D, E) edge if G'^R falls under case A (B, C, D, E) of the present discussion, where G' is again the graph formed from G by removing all extra-skeletal edges other than $i_r j_r$.

Case D $a_1 \leq j_1 < b_1$ and $a_2 < i_1$. If $a_1 = j_1$, then $i_1 < b_2$, otherwise the edge $i_1 j_1$ taken with the path P forms a cycle through all the vertices of G, a contradiction. However now $i_1 j_1$ and $a_2 a_1$ can interchange roles, and we are in case B. Thus in the present case, we assume without loss of generality that $a_1 < j_1$. Symmetrically, we assume that $i_1 < b_2$, or G is the reverse of a graph falling under case B. A reduction of G can be mimicked by MAX.16. (See Figure 5.6)

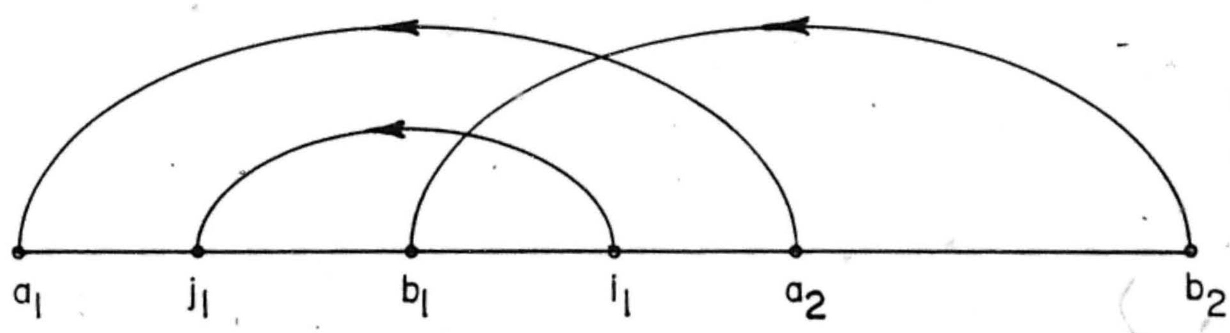


Figure 5.5

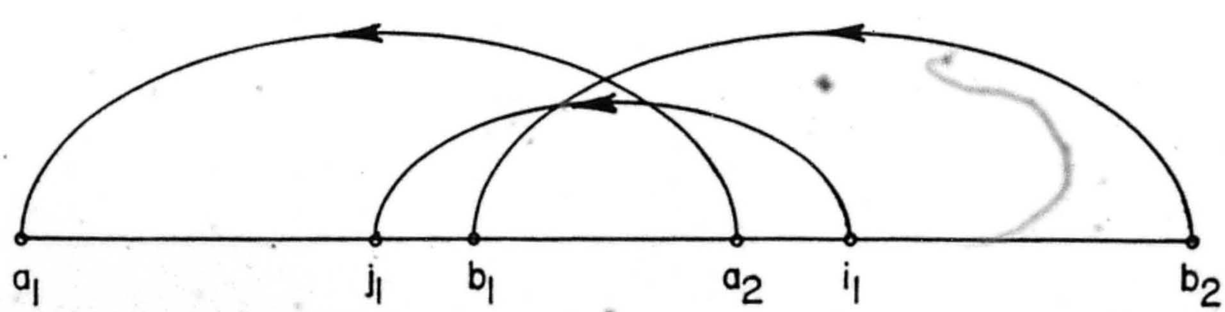


Figure 5.6

This concludes an enumeration of the subcases when $a_1 \leq j_1 < b_1$.

To reduce work, we again invoke symmetry. Let G have an edge i_1j_1 with $a_2 < i_1 \leq b_2$. Then G^R, the reverse of G, is clearly a two hump digraph. Again, vert(G^R) = vert(P^R) where P^R is the reverse of P. Renaming a_1 as b_2', a_2 as b_1', b_1 as a_2', b_2 as a_1', i_1 as j_1' and j_1 as i_1', we see that G^R is a two hump digraph with skeleton $P^R \cup \{ b_2'b_1', a_2'a_1' \}$ and an additional edge $i_1'j_1'$ with $a_1' \leq j_1 < b_1'$. We see that the case when $a_2 < i_1 \leq b_2$ and the case when $a_1 \leq j_1 < b_1$ are symmetric and may be regarded as equivalent. For m > 1, however, it will occasionally be necessary to distinguish between "normal" type B or C edges, and " reversed " type B or C edges.

Case E $b_1 \leq j_1 \leq a_2$ (and $i_1 \leq a_2$).

Here the edge i_1j_1 is useless, a contradiction. (See Figure 5.7)

Thus when m = 1, and i_1j_1 is a back edge, G is mimicked by one of the graphs of MAX. We also draw attention to the 5 basic types of back edge which G can have. These 5 types of edges figure in later case

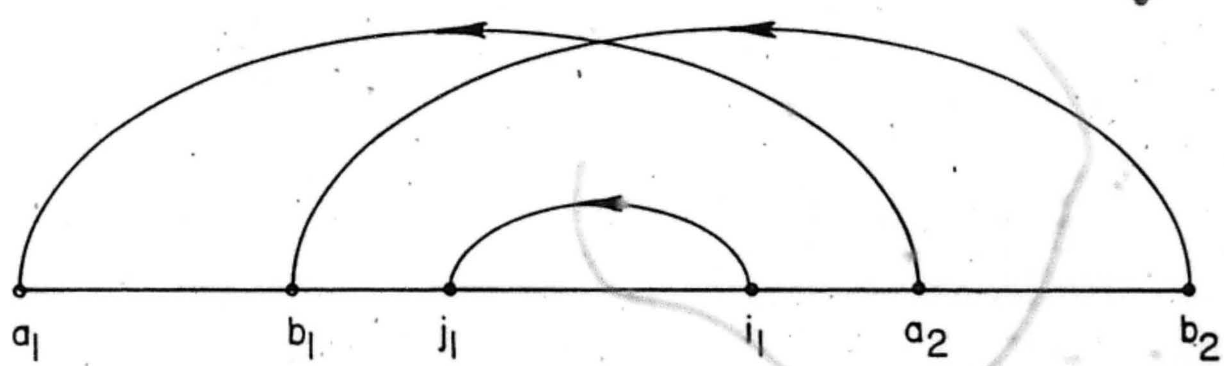


Figure 5.7

divisions.

CASE II (ONE FORWARD EDGE)

Case II also gives rise to several subcases; however, this case may be dealt with very simply once we have made the following observation: For most forward edges ij, a skeleton for G can be chosen so that ij becomes a back edge with respect to that skeleton.

As we remarked earlier, there is a skeleton for G in which the roles of X and Y are reversed. Now the vertices of G may be divided into three sets:

$vert(X \setminus Y)$, $vert(X \cap Y)$, $vert(Y \setminus X)$.

With respect to the skeleton we have given for G, these sets occur in this order. However, if the roles of X and Y are reversed, then the order of these sets reverses. Thus if i_1 and j_1 are not both in the same one of these sets, a forward edge i_1j_1 becomes a back edge when X and Y are interchanged. Therefore in the present case we assume without loss of generality that i_1 and j_1 are both in the same one of the listed sets.

Case II.1: We have $i_1, j_1 \in X \setminus Y$, i.e.

$a_1 \leq i_1 < j_1 < b_1$.

Some vertex x of P must lie between i_1 and j_1. Here G mimics MIN.2 (See Figure 5.8)

Case II.2: We have i_1, $j_1 \in X \cap Y$, i.e., $b_1 \le i_1 < j_1 \le a_2$.
Consider the cycle C differing from X in that the path from i_1 to j_1 in X is replaced in C by the edge $i_1 j_1$. (See Figure 5.9.) Then cycles C and Y have an intersection which is not connected, and by the proof of the Intersection Lemma, Lemma 3.2, G mimics MIN.1 or MIN.2

Case II.3: We have i_1, $j_1 \in Y \setminus X$.
Interchanging the roles of X and Y, this case is equivalent to case II.1.

This concludes CASE II, and hence the case when m = 1.

m > 1:

In light of the foregoing, we may now assume that any individual forward edge of G may be turned into a back edge by interchanging cycles X and Y. We will now show that in fact all extra-skeletal edges of G may simultaneously be assumed to be back edges. Suppose that $i_1 j_1$ is a back edge and $i_2 j_2$ is a forward edge. By Case II, we may assume that reversing the roles of X and Y

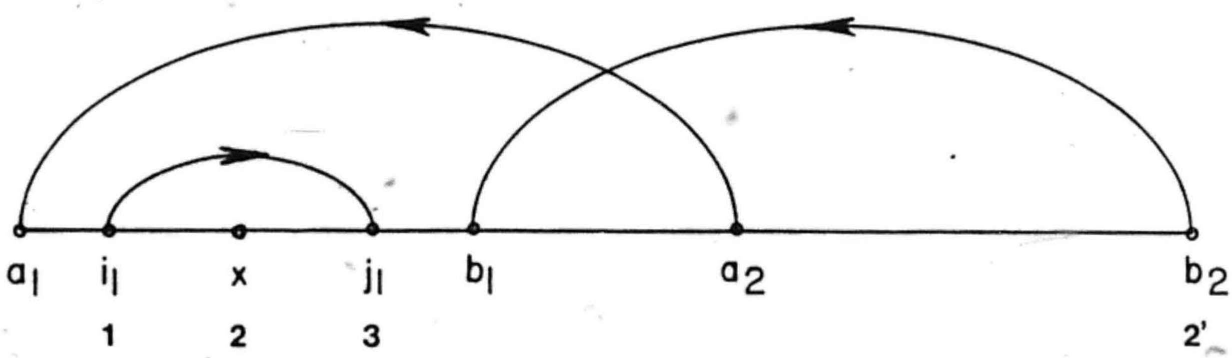


Figure 5.8

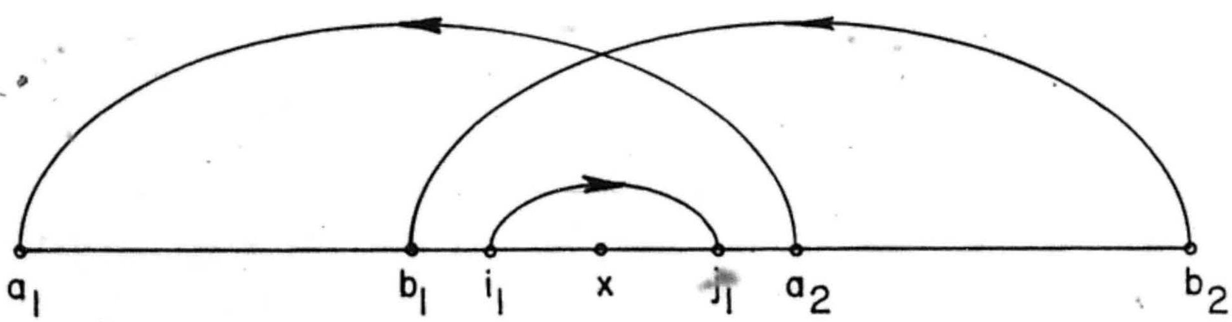


Figure 5.9

would make i_2j_2 a back edge. We make cases as follows.

Case II+A: Edge i_1j_1 falls under case A. By assumption, edge i_2j_2 becomes a back edge when the roles of X and Y are interchanged. However, in the present case, when X and Y are switched, i_1j_1 remains a back edge (since i_1, $j_1 \in X$). We may thus assume that both i_1j_1 and i_2j_2 are back edges.

Case II+B: Edge i_1j_1 falls under case B.

Case II+C: Edge i_1j_1 falls under case C.

Case II+D: Edge i_1j_1 falls under case D.

Case II+E: Edge i_1j_1 falls under case E. In this case, if the roles of X and Y are interchanged, i_1j_1 and i_2j_2 are both back edges and we are finished. (Both i_1, $j_1 \in X \cap Y$.)

In case II+B we make the following subcases based on i_2j_2:

Case II+B 1: $a_1 \le i_2 < b_1$.

Case II+B 1 (a) $b_1 \le j_2 \le i_1$. Consider the cycle C differing from X in that the path from i_2 to j_2 in X is replaced in C by the edge i_2j_2 and the path from i_1 to a_1 in X is replaced by the edge i_1a_1. (See Figure 5.10) Then cycles C and X have disconnected intersection, and by the proof of the intersection lemma,

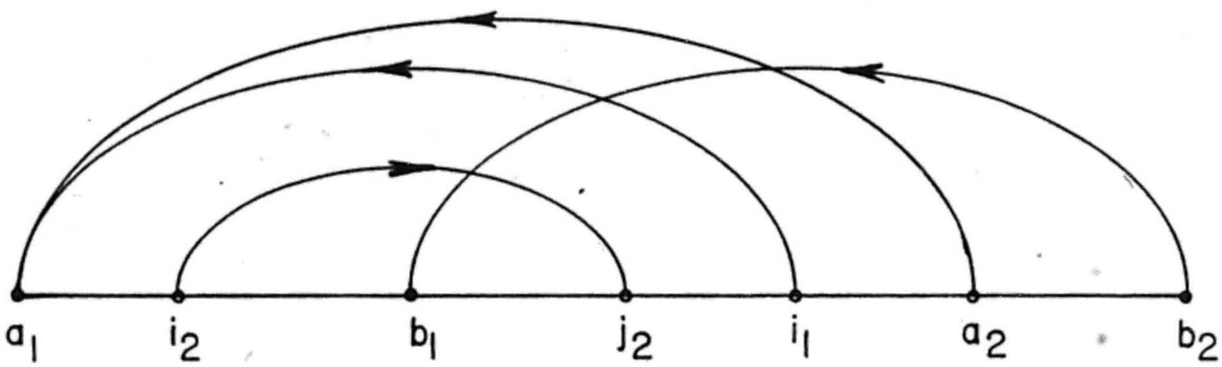


Figure 5.10

G mimics MIN.1 or MIN.2.

Case II+B 1 (b): $i_1 < j_2$. Consider the cycle C following P from a_1 to i_2, then edge i_2j_2, then P to b_2, then b_2b_1, then P from b_1 to i_1, finally i_1a_1. (See Figure 5.11) Then cycles C and X have disconnected intersection, and by the intersection lemma, G mimics MIN.1 or MIN.2.

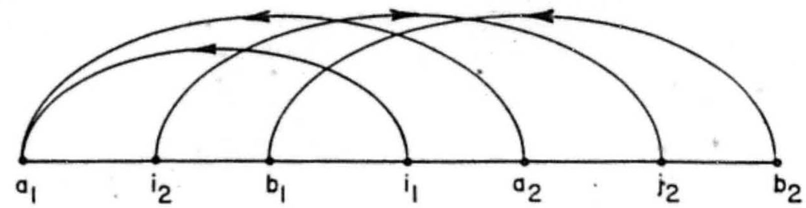
Case II+B 2: $b_1 \leq i_2 < a_2$. Since $i_2 \in X \cap Y$, assume that $j_2 \in Y \setminus X$, viz. $a_2 < j_2$. Then G mimics the triangle, MIN.1. (See Figure 5.12)

In case II+C we make the following subcases based on i_2j_2:

Case II+C 1: $a_1 \leq i_2 < j_1$. (Thus $b_1 \leq j_2$.) Again G mimics the triangle. (See Figure 5.13.)

Case II+C 2: $j_1 \leq i_2 < b_1$. Again, $b_1 \leq j_2$.

Case II+C 2 (a): $b_1 \leq j_2 \leq i_1$. Consider the cycle C following P from j_1 to i_2, then edge i_2j_2, then P to i_1, finally i_1j_1. (See Figure 5.14). Then cycles C and X have disconnected intersection, and by the intersection lemma, G mimics MIN.1 or MIN.2.



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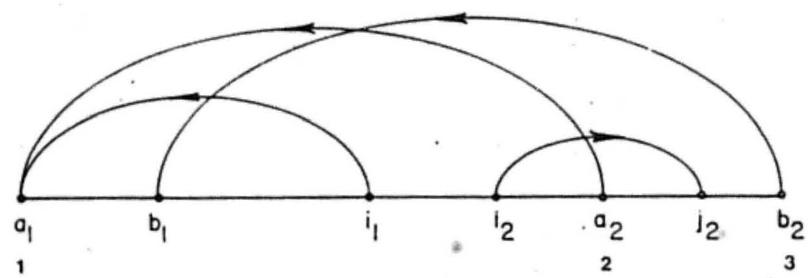
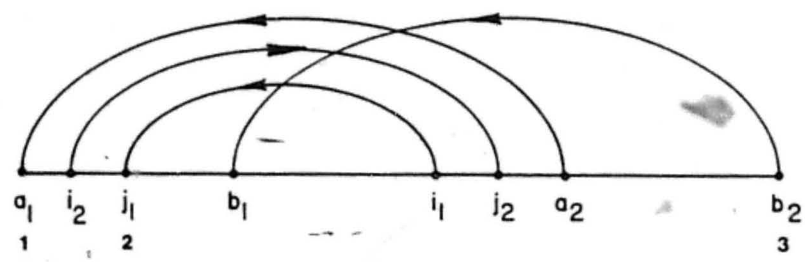


Figure 5.12



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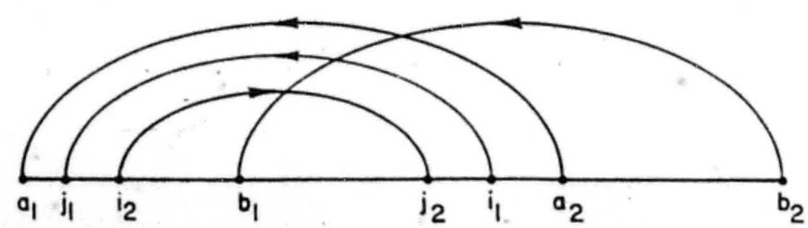


Figure 5.14

Case II+C 2 (b) $i_1 < j_2$. Consider the cycle C following P from j_1 to i_2, then edge i_2j_2, then P to b_2, then b_2b_1, then P from b_1 to i_1, finally i_1j_1. (See Figure 5.15). Then cycles C and X have disconnected intersection, and by the intersection lemma, G mimics MIN.1 or MIN.2.

Case II+C 3: $b_1 \le i_2 < a_2$. (Thus $a_2 < j_2$.). We have two cases:

(i) $i_1 < a_2$. Here G mimics the triangle. (See Figure 5.16)

(ii) $i_1 = a_2$. Here G mimics MIN.22. (See Figure 5.17.)

Case II+C 4: $i_2 = a_2$. Here G mimics MIN.2 (See Figure 5.18)

We can use case II+C to attack case II+D. Suppose that G has an edge ij of type D. Then G can mimic a digraph H which falls under case C. (See Figure 5.19) One walks 32 modulo paths on G by following P from a_2 to i, then edge ij. Similarly, one walks 43 by following edge b_2b_1, then P from b_1 to a_2.

Figure 5.15

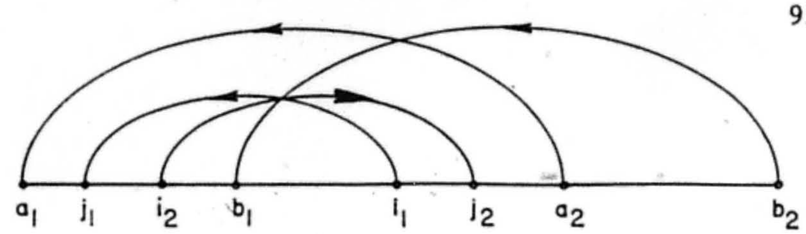


Figure 5.16

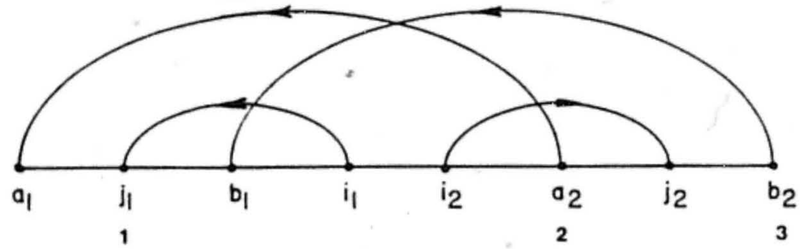


Figure 5.17

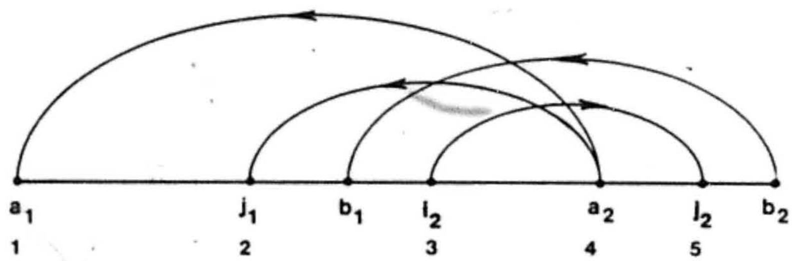


Figure 5.18

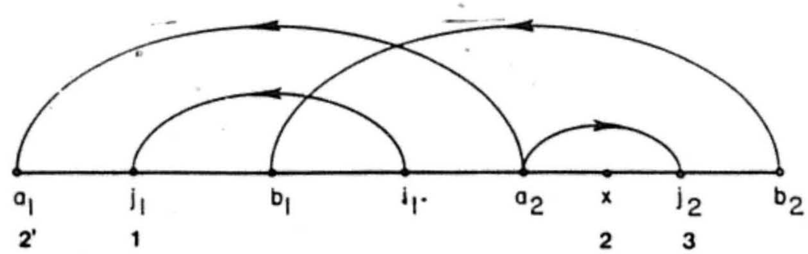
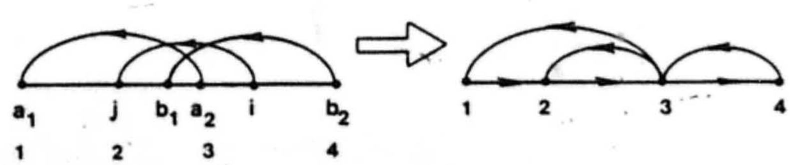


Figure 5.19



Suppose that G also has a forward edge kl. Then usually, G can mimic a digraph H' derived from H by adding a forward edge. Digraph H' will fall under case II + C, and hence be versatile. An example is shown in Figure 5.20. Difficulties only arise if $b_1 \leq k$, $l \leq i$. In such a case, each of k and l lies on one of the paths on which we would walk edges 32 and 43 of H.

Suppose then that $b_1 \leq k$, $l \leq i$. By the symmetry of type D edges under reflection, we assume that $b_1 \leq k$, $l \leq a_2$. However this means that both k and l lie in $X \cap Y$, and this case was dealt with under II.1.

We have now shown that i_1j_1 and i_2j_2 may both be assumed to be back edges. A simple induction on the number of forward edges in G shows that we may assume that every extraskeletal edge of G is a back edge.

For economy of cases in the rest of this chapter, we will use the following case divisions:

(1) Every extraskeletal edge of G is a back edge of case E.

(2) Graph G has a back edge of case A and (up to reversal) every back edge of G is a back edge of either case A or E.

(3) Graph G has a back edge of case B and (up to

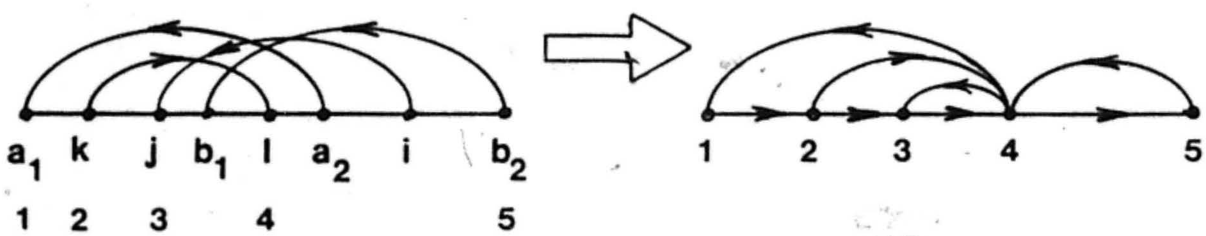


Figure 5.20

reversal) every back edge of G is a back edge of either case B, A, or E.

(4) Graph G has a back edge of case C and (up to reversal) every back edge of G is a back edge of either case C, B, A or E.

(5) Graph G has a back edge of case D.

<u>Case (1)</u>: Every extraskeletal edge of G is a back edge of case E.

If every edge of G is of type E then without loss of generality i_1j_1 and i_2j_2 form an M where $a_1 < b_1 \leq j_1 < j_2 \leq i_1 < i_2 \leq a_2 < b_2$. Here G mimics MIN.15. (See Figure 5.21)

<u>Case (2):</u> Graph G has a back edge of case A and (up to reversal) every back edge of G is a back edge of either case A or E.

Note that an edge of type A can never form an M with an edge of type E. Thus if G has an edge of type E, it will have two type E edges forming an M as in the previous case. Therefore we may assume in this case that G has only type A back edges.

If m = 2, then without loss of generality (invoking

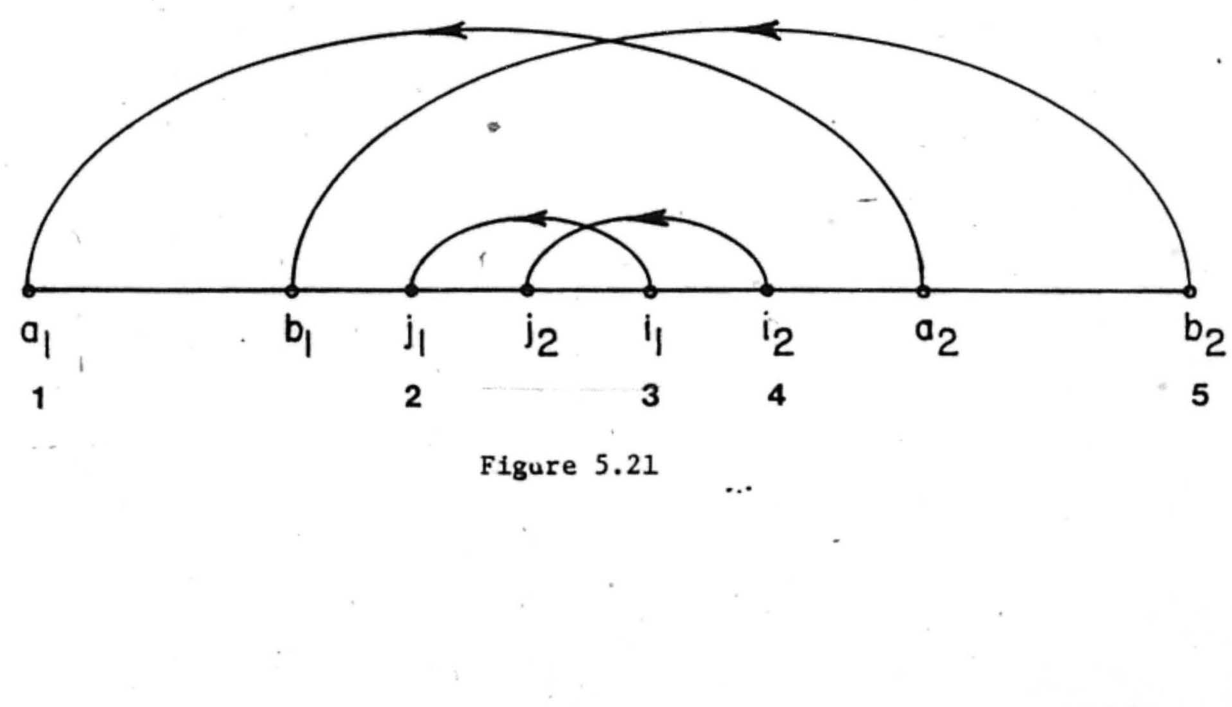


Figure 5.21

the M lemma, and renaming if necessary) $a_1 \leq j_1 < j_2 \leq i_1 < i_2 < b_1$.

If $a_1 = j_1$, then a reduction of G is mimicked by MAX.7. (See Figure 5.22)

If b_1 is the successor of i_2 on P, then a reduction of G is mimicked by MAX.15. (See Figure 5.23)

If $a_1 < j_1$ and there is a vertex of G between i_2 and b_1, then G mimics MIN.18. (See Figure 5.24.)

For the remainder of Case (2), assume that $m > 2$.

Suppose that G has edges $i_1 j_1$, a type A edge, and $i_2 j_2$, a type A edge after reversal, i.e. $a_1 \leq j_1 < i_1 < b_1$ and $a_2 < j_2 < i_2 \leq b_2$. Then without loss of generality, G also has edges $i_3 j_3$ and $i_4 j_4$ where $a_1 \leq j_3 < j_1 \leq i_3 < i_1 < b_1 \leq a_2 < j_4 < j_2 \leq i_4 < i_2 \leq b_2$. Here G mimics MIN.19. (See Ffgure 5.25)

We may thus assume that every extra-skeletal edge $i_s j_s$ of G is a true type A edge, viz. $a_1 \leq j_s < i_s < b_1$. We now introduce a "stripping" method of classification, that will serve us again in the next chapter. Since G has only type A edges, and these only in the first "half" of G, we strip away other edges of G, and use these type A edges for our classification:

Let G' be the graph obtained from G by removing the

Figure 5.22

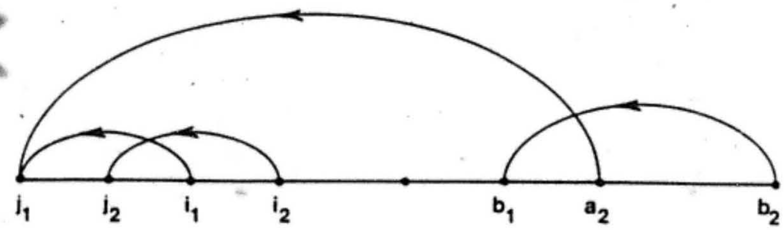


Figure 5.23

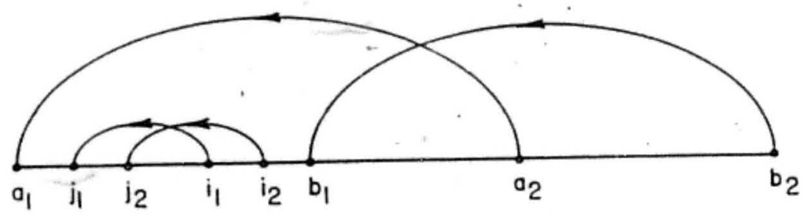


Figure 5.24

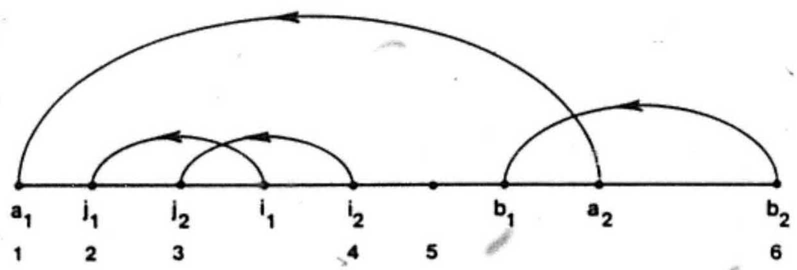
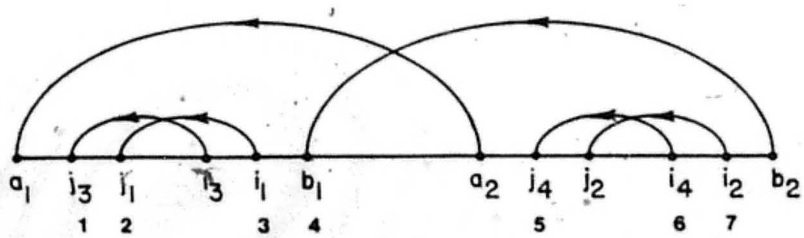


Figure 5.25



edges a_2a_1 and b_2b_1. Consider the strongly connected components of G' consisting of more than one vertex. At least one such component exists, since G has back edges. If more than one such component exists, then without loss of generality G has edges i_1j_1, i_2j_2, i_3j_3, i_4j_4 where

$$a_1 \le j_1 < j_2 \le i_1 < i_2 < j_3 < j_4 \le i_3 < i_4 < b_1,$$

since each of these components contains an M. Here G mimics MIN.20. (See Figure 5.26)

Thus, without loss of generality, we may speak of <u>the</u> strongly connected component G" of G' containing more than one vertex. Since G" is a strongly connected digraph, we may invoke our previous classification results to say things about the structure of G". This is our "stripping" method.

<u>Case- G" is a graph of type (3) of the Classification Lemma (Lemma 3.3):</u>

Without loss of generality G has three back edges i_1j_1, i_2j_2, i_3j_3, with $a_1 \le j_1 < j_2 \le i_1 < j_3 \le i_2 < i_3 < b_1$. Here G mimics MIN.21.

(See Figure 5.27)

<u>Case- G" is a graph of type (2) of the classification lemma:</u>

<u>Subcase- G" has edges of type E only</u>: As we

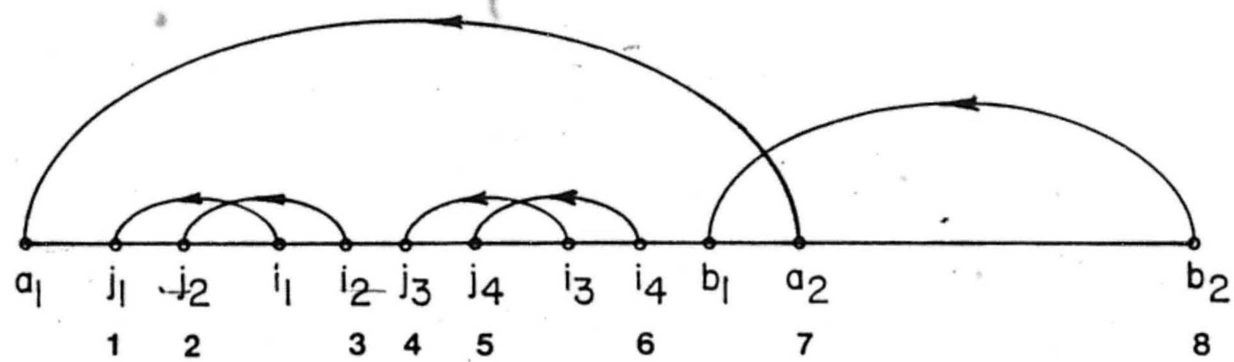


Figure 5.26

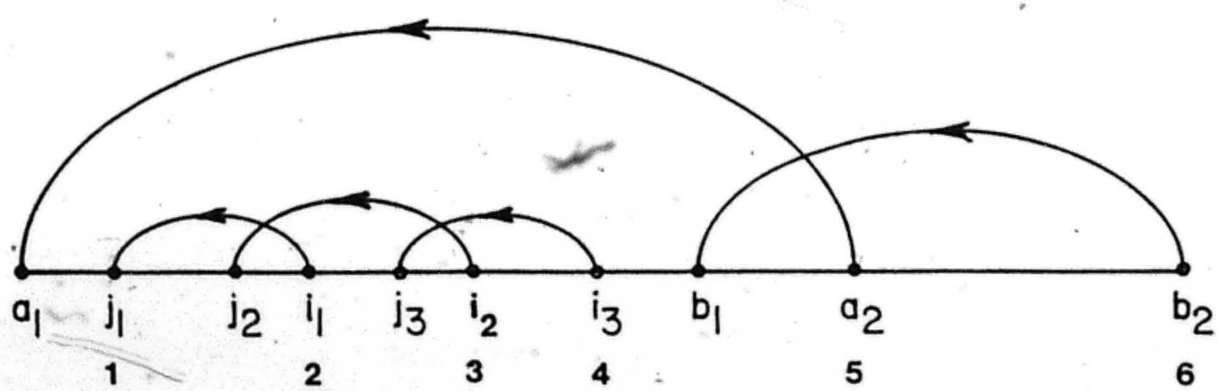


Figure 5.27

have seen previously, G" , hence G, mimics MIN.15.

<u>Subcase- G" has edges of type A only:</u> Here without loss of generality (up to reversing the roles of X and Y and reversal) G has edges i_1j_1, i_2j_2, i_3j_3, i_4j_4 with

$a_1 \le j_1 \le j_2 < j_3 \le i_2 < i_3 < j_4 \le i_1 < i_4 < b_1$. Here G mimics MIN.66.(See Figure 5.28)

<u>Subcase- G" has an edge of type B:</u> Here without loss of generality (up to reversing the roles of X and Y and reversal) G has edges i_1j_1, i_2j_2, i_3j_3 with

$a_1 \le j_1 = j_3 < j_2 \le i_3 < i_1 < i_2 < b_1$. Here G mimics MIN.23. (See Figure 5.29)

<u>Subcase- G" has an edge of type C:</u> Here without loss of generality (up to reversing the roles of X and Y and reversal) G has edges i_1j_1, i_2j_2, i_3j_3 with

$a_1 \le j_1 < j_3 < j_2 \le i_3 \le i_1 < i_2 < b_1$. Here G mimics MIN.24.(See Figure 5.30)

<u>Subcase- G" has an edge of type D:</u> Here without loss of generality (up to reversing the roles of X and Y) G has edges i_1j_1, i_2j_2, i_3j_3 with

$a_1 \le j_1 < j_3 < j_2 \le i_1 < i_3 < i_2 < b_1$. Here G mimics

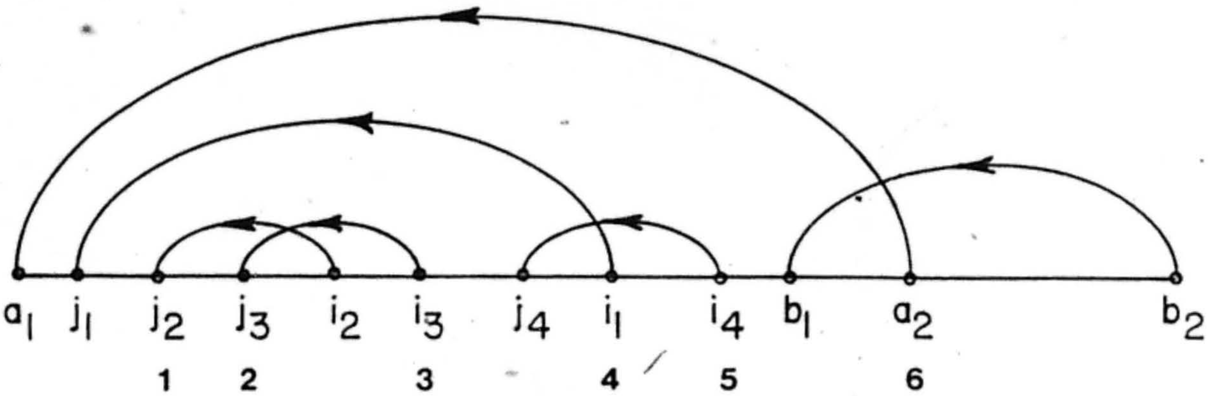


Figure 5.28

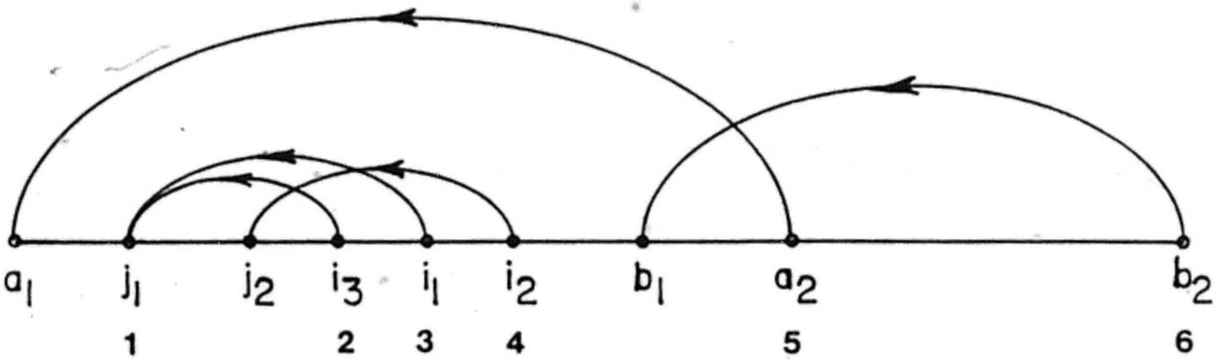


Figure 5.29

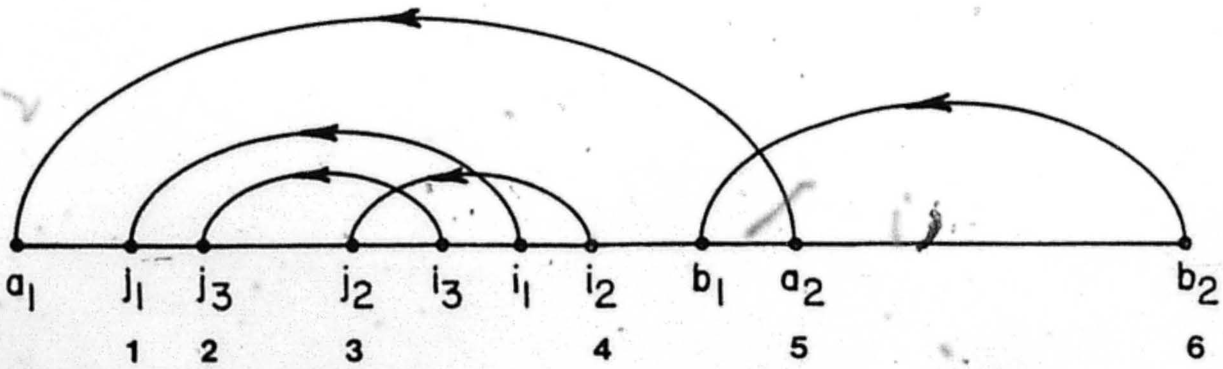


Figure 5.30

MIN.24.(See Figure 5.31)

Thus if G" falls under type (2) of the classification lemma, G mimics a digraph of MIN.

<u>Case- G" is a graph of type (1) of the classification lemma:</u>

Here without loss of generality G has edges i_1j_1, i_2j_2, i_3j_3 with $a_1 \leq j_3 \leq j_2 < j_1 \leq i_2 < i_1 < i_3 < b_1$. (The edge i_3j_3 is the back edge of the skeleton of G". Since i_3j_3 can be assumed to be a useful edge, we pick i_1j_1, i_2j_2 to form an M under i_3j_3. Thus either $j_3 \neq j_2$, or $i_1 \neq i_3$. We may assume that $i_1 \neq i_3$ without loss of generality, up to reversal of G, or the interchanging of cycles X and Y.)

We form subcases:

<u>Subcase- $a_1 < j_2$:</u> Here G mimics MIN.18. (See Figure 5.32)

<u>Subcase- $j_2 = a_1 = j_3$, $m = 3$:</u> In this case a reduction of G can be mimicked by MAX.14.(See Figure 5.33)

Subcase-<u>$m > 3$</u>: Either repeating our stripping process on G" will lead to a graph of type (3) of the

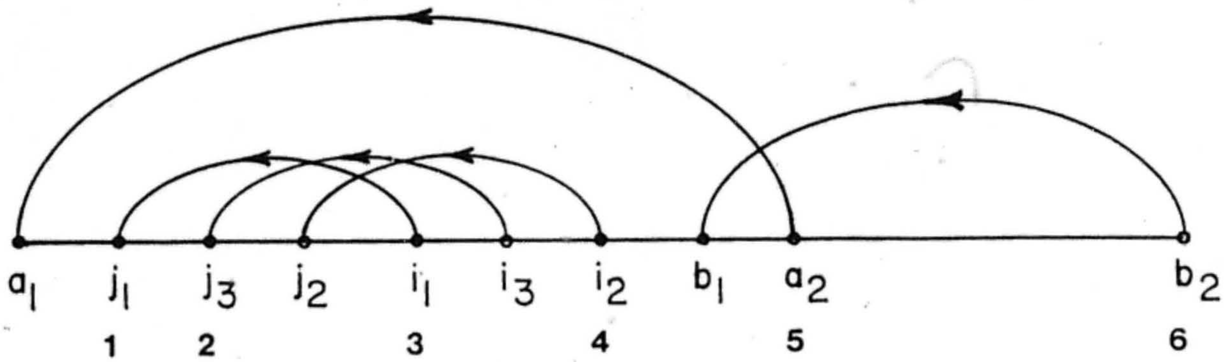


Figure 5.31

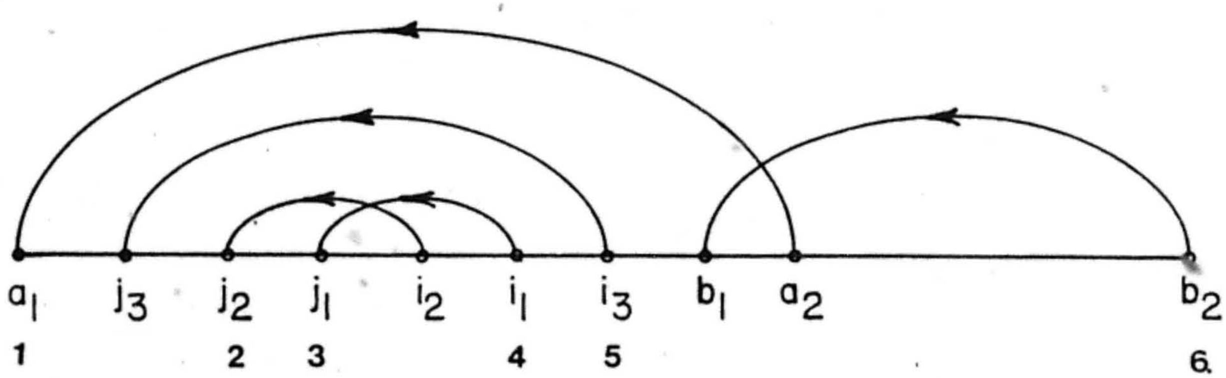


Figure 5.32

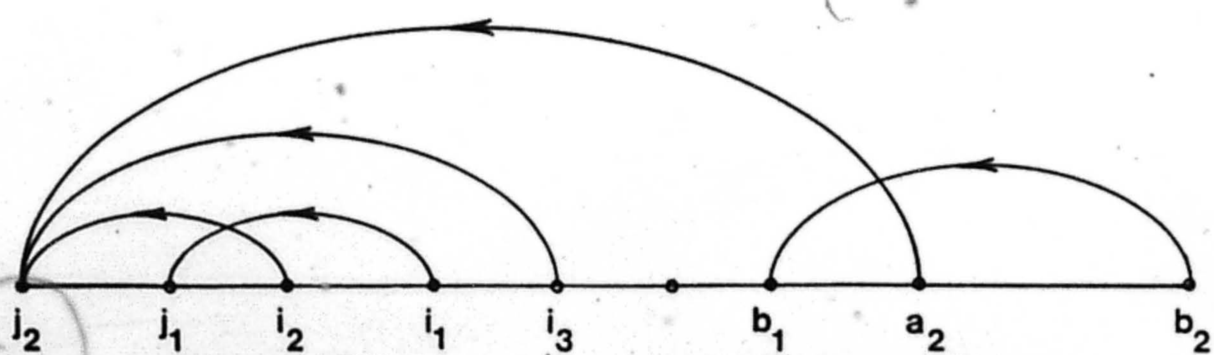


Figure 5.33

classification lemma, or a graph of type (2) of the classification lemma with a back edge, or a graph of type (1) of the classification lemma.

We may assume, without loss of generality that the first two cases do not occur. Assume without loss of generality that $a_1 = j_2 = j_3$. Otherwise G mimics a digraph of MIN as already shown above. Without loss of generality, up to reversal of G, or the interchanging of cycles X and Y, G contains an edge i_4j_4 with $j_4 = a_1$ and $i_1 < i_4 < i_3$. Thus G mimics MIN.25. (See Figure 5.34)

This completes our examination of Case (2).

<u>Case (3):</u> Graph G has a back edge of type B and (up to reversal) every back edge of G is a back edge of either type B, A, or E.

In analogy to the previous case, we first dismiss the cases where not every edge of G is a type B edge.

Suppose that G has an edge i_2j_2 of type E. If the edge of G intersecting i_2j_2 to form an M is a type E edge, then we are done, as in case (1). Thus without loss of generality we may assume that

$$a_1 = j_1 < b_1 \leq j_2 \leq i_1 < i_2 \leq a_2 < b_2.$$

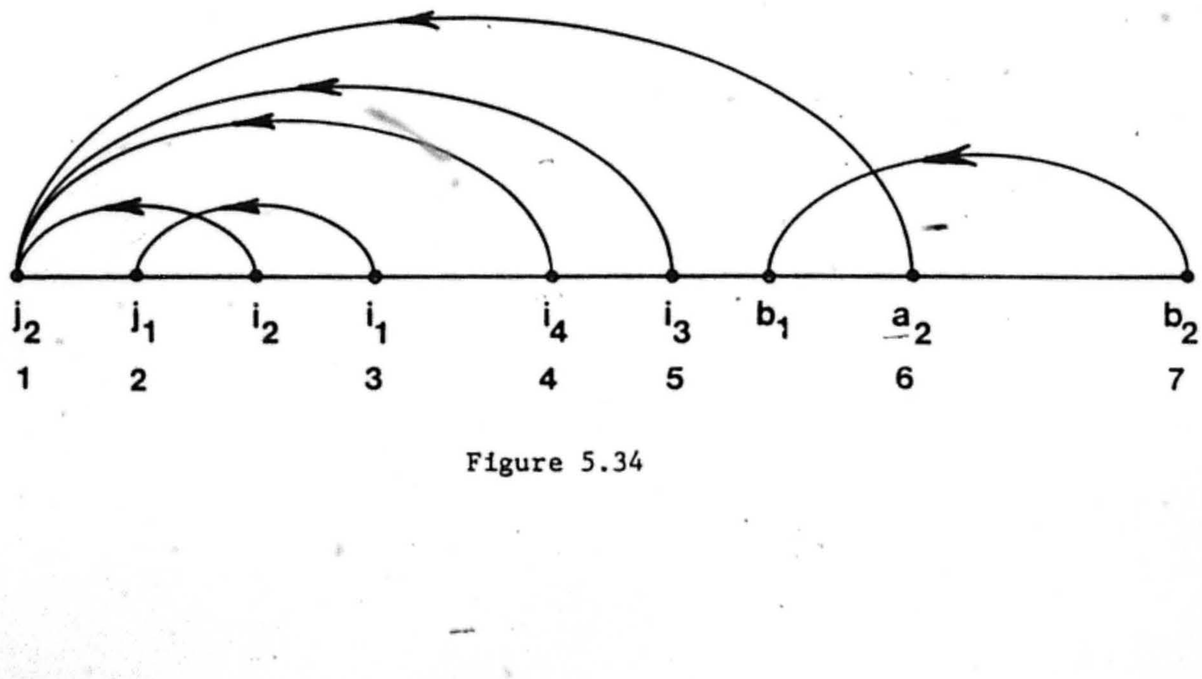


Figure 5.34

If we now interchange the roles of edges i_1j_1 and a_2a_1, then with respect to the new skeleton for G, i_2j_2 is an edge of type C. (See Figure 5.35). We may thus delay discussion of this possibility until case (4).

From now on, let us assume that G has no edges of type E.

Suppose that G has a type A edge i_2j_2. Then without loss of generality, i_2j_2 forms an M with a type A edge i_3j_3. We have two possibilities:

(i) $a_1 = j_1 \leq j_2 < j_3 \leq i_2 < i_3 < b_1 \leq i_1 < a_2$. Here G mimics MIN.26. (See Figure 5.36)

(ii) $a_1 \leq j_2 < j_3 \leq i_2 < i_3 < b_1 < j_1 \leq a_2 < i_1 = b_2$. Here i_1j_1 is a reversed type B edge. Here G mimics MIN.27. (See Figure 5.37)

For the remainder of this case we assume that G has (up to reversal) only type B edges. For convenience, we rename edges here:

Let i_1j_1, i_2j_2, ..., i_rj_r be the (normal) type B edges of G

$j_1 = j_2 = \ldots = j_r = a_1 < b_1 \leq i_1 < i_2 < \ldots < i_r < a_2$.

Let k_1l_1, k_2l_2, ..., k_sl_s be the (reversed) type B

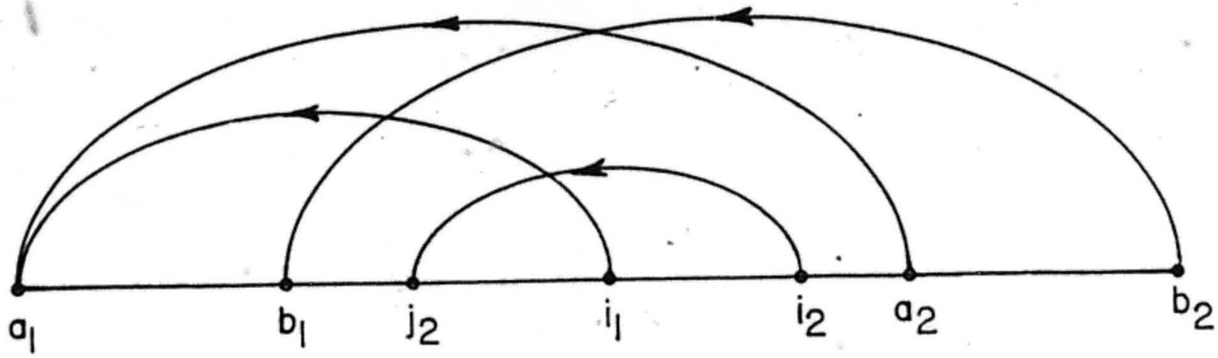


Figure 5.35

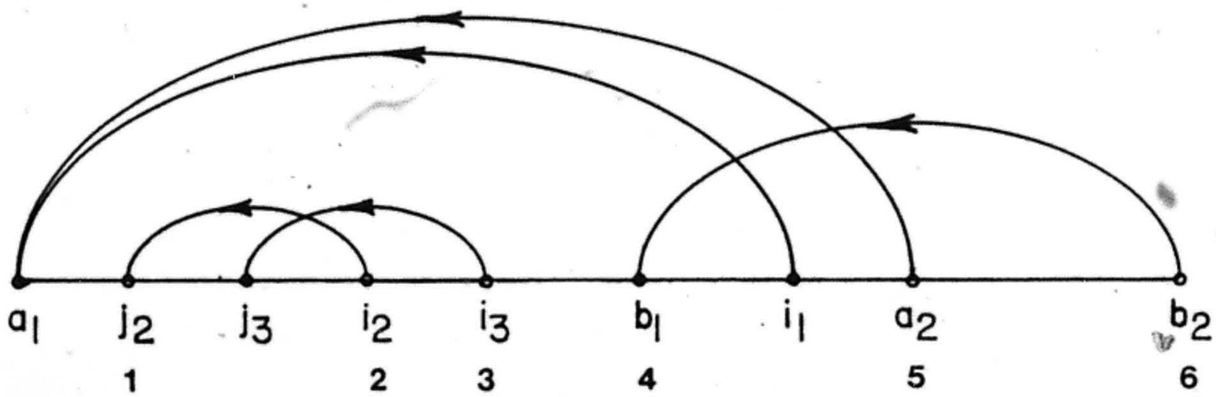


Figure 5.36

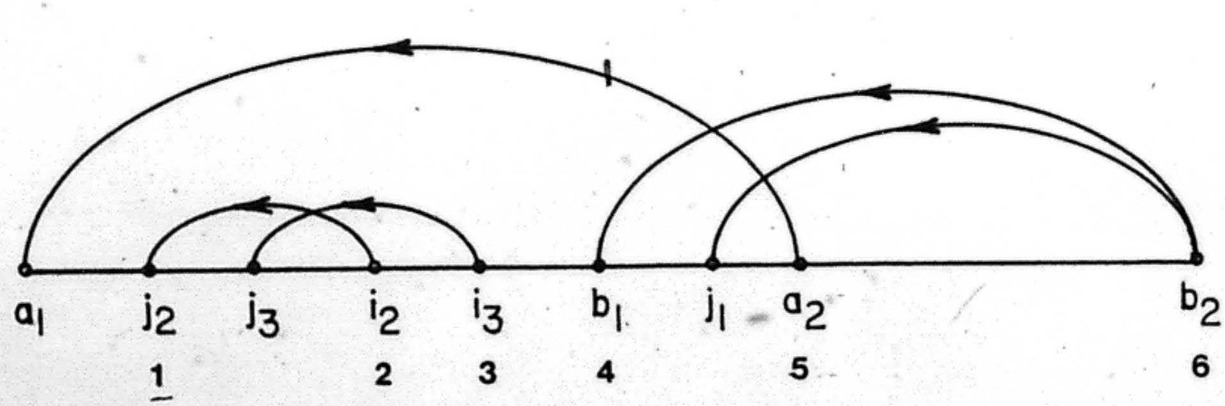


Figure 5.37

edges of G

$b_1 < l_1 < l_2 < \ldots < l_s \leq a_2 < k_1 = k_2 = \ldots = k_s = b_2$.

If s = 0 then for large enough q, a reduction of G is mimicked by MAX.13. (See Figure 5.38)

If s > 0 and for some t and u, $l_t \leq i_u$, then G mimics MIN.28. (See Figure 5.39)

We may thus assume from now on that $r \geq s > 0$, and $l_t > i_u$ for all t, u where $1 \leq u \leq r$, $1 \leq t \leq s$.

Our remaining subcases are based on the values of s and r.

<u>Subcase r < 3, s = 1:</u> Here a reduction of G is mimicked by MAX.12. (See Figure 5.40)

<u>Subcase r = 2, s > 1:</u> Here G mimics MIN.29. (See Figure 5.41.).

<u>Subcase r > 2:</u> Here G mimics MIN.30. (See Figure 5.42.)

This concludes our examination of case (3).

It will prove economical to deal with case (5) before

Figure 5.38

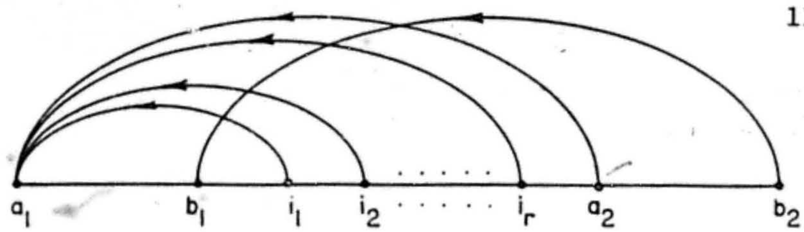


Figure 5.39

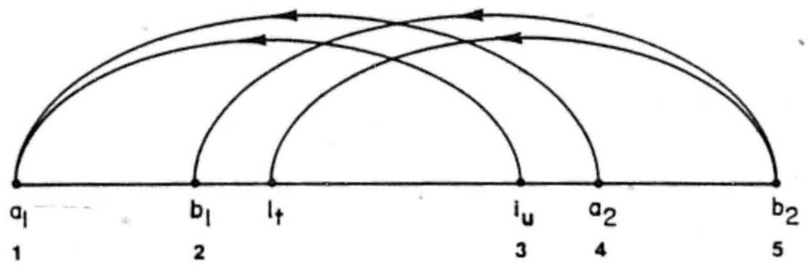


Figure 5.40

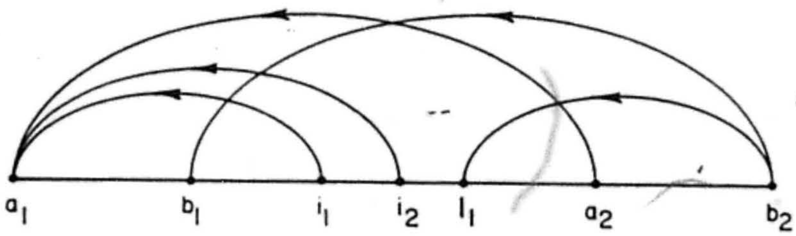


Figure 5.41

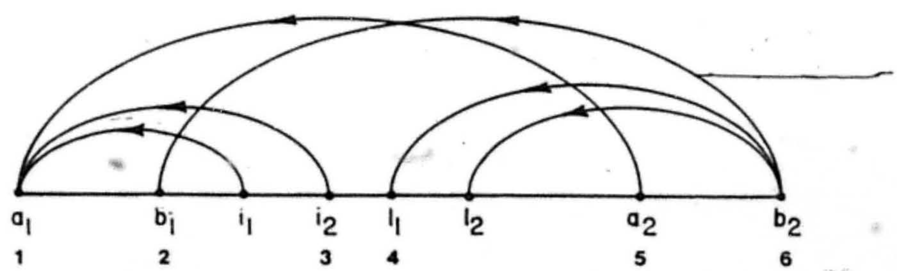
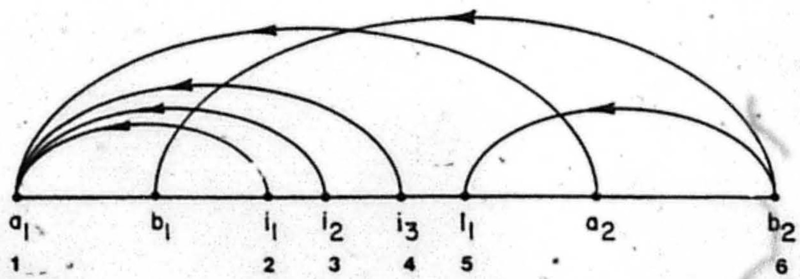


Figure 5.42



case (4).

<u>Case (5):</u> Graph G has a back edge of case D.

Note that case (5) shows mirror symmetry; If G is reversed, then switching the roles of a_1, a_2, b_1, b_2, i_1, j_1 with b_2, b_1, a_2, a_1, j_1, i_1 respectively again gives us case (5). This symmetry allows us to reduce our number of cases.

<u>m = 2:</u>

We make the following case divisions based on j_2:

Case Dα: $j_2 < j_1$.

Case Dβ: $j_1 \leq j_2 < b_1$ ($i_2 \leq i_1$ by symmetry).

Case Dγ: $b_1 \leq j_2 \leq a_2$ ($i_2 \leq a_2$). In this case the edge i_2j_2 is useless, a contradiction.

(See Figure 5.43)

Case Dα is subdivided as follows depending on i_2:

Case Dα1: $i_2 < j_1$. In this case the edge i_2j_2 is useless, a contradiction. (See Figure 5.44)

Case Dα2: $j_1 \leq i_2 < i_1$. We have two possibilities:

(i) $i_2 \neq a_2$. Here G mimics MIN.31.

(See Figure 5.45. Here the greater of a_2, i_2 is labelled

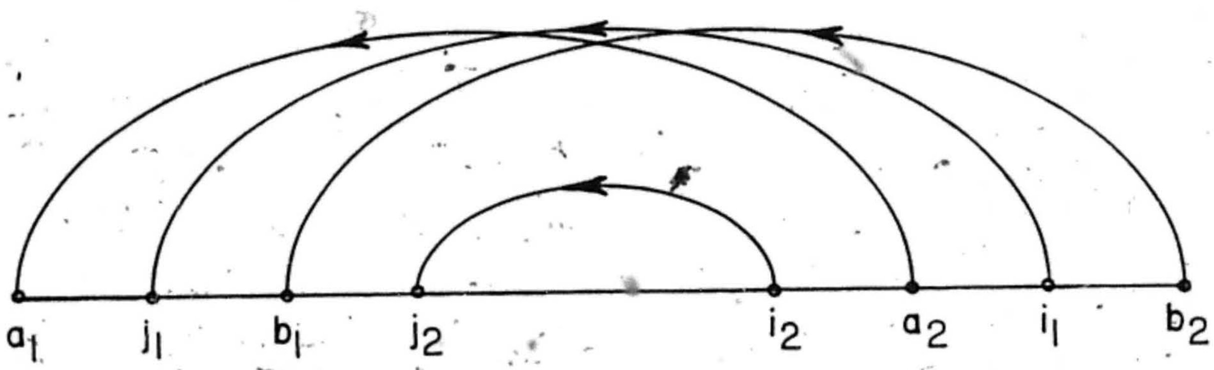


Figure 5.43

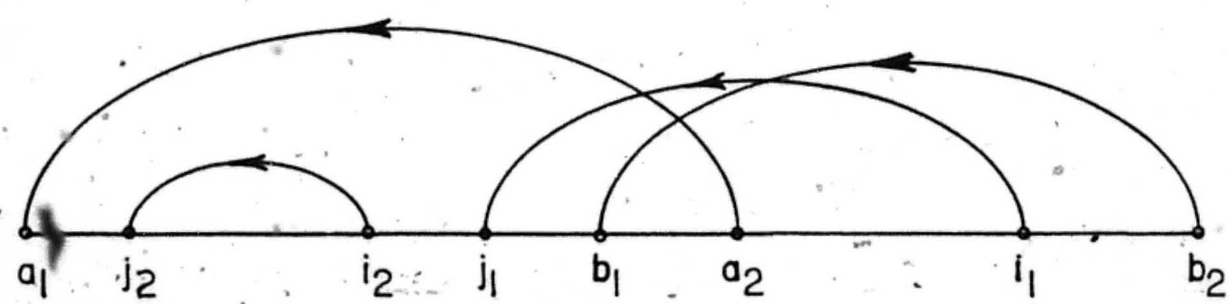


Figure 5.44

3.)

(ii) $i_2 = a_2$. Here G mimics MIN.32. (See Figure 5.46)

Case Dα3: $i_1 \leq i_2$. (Thus either $i_1 < i_2$, or $i_2 < b_2$. Note that if $j_2 = a_1$ <u>and</u> $i_2 = b_2$, then G is a one hump digraph. By symmetry, assume that $i_2 < b_2$.) Here G mimics MIN.33. (See Figure 5.47.)

In Case Dβ, we may assume that $i_2 < i_1$, otherwise interchanging the roles of i_1j_1, i_2j_2 gives Case Dα. Case Dβ is subdivided as follows depending on i_2:

Case Dβ1: $i_2 < b_1$. In this case the edge i_2j_2 is useless, a contradiction. (See Figure 5.48)

Case Dβ2: $b_1 \leq i_2$. Here G mimics MIN.32. (See Figure 5.49)

This concludes the subcase when m = 2.

<u>m > 2:</u>

By the foregoing analysis we may assume that any extra-skeletal edge of G falls (up to reversal) under cases Dα1, Dβ1 or Dγ.

Figure 5.45

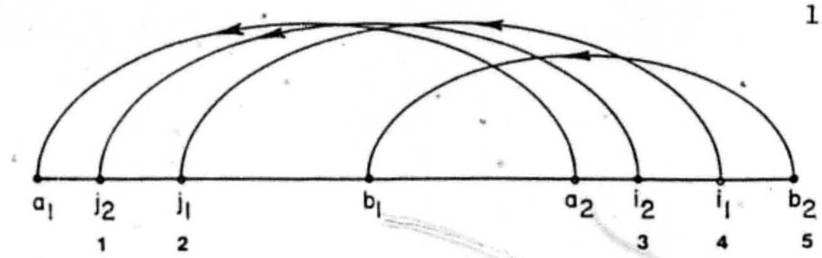


Figure 5.46

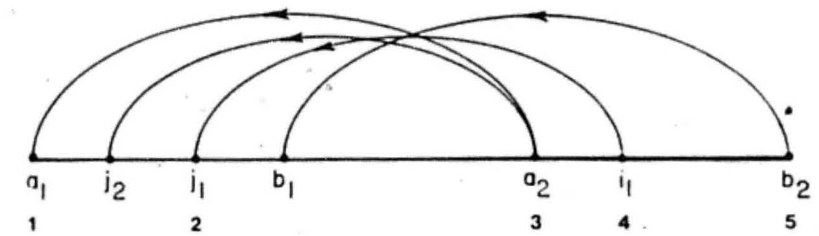


Figure 5.47

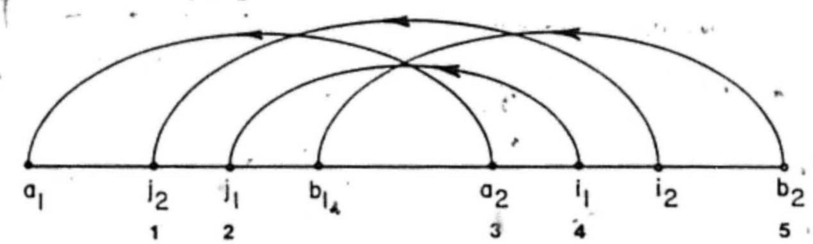


Figure 5.48

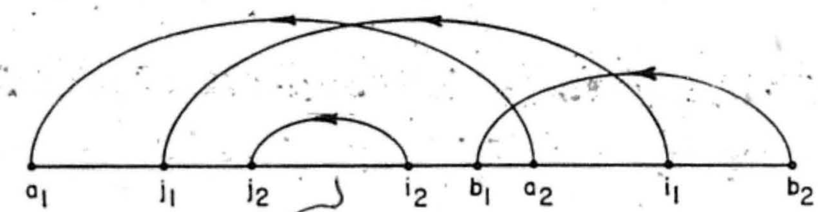
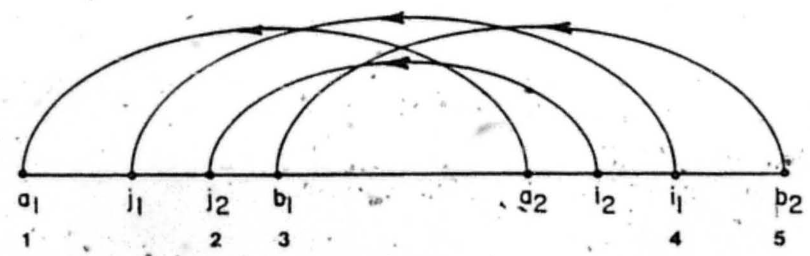


Figure 5.49



<u>Case- Edges of type Dα1:</u> Without loss of generality, invoking the M lemma, G has edges i_2j_2, i_3j_3, with $a_1 \leq j_2 < j_3 \leq i_2 < i_3 < j_1$. In this case, G mimics MIN.14. (See Figure 5.50)

<u>Case- Edges of type Dβ1:</u> Without loss of generality, G has edges i_2j_2, i_3j_3, with $j_1 \leq j_2 < j_3 \leq i_2 < i_3 < b_1$. In this case, G mimics MIN.15. (See Figure 5.51)

<u>Case- Edges of type Dγ:</u> Without loss of generality, G has edges i_2j_2, i_3j_3, with $b_1 \leq j_2 < j_3 \leq i_2 < i_3 \leq a_2$. In this case, G mimics MIN.15, as in case (1). (See Figure 5.52)

This concludes our examination of case (5).

<u>Case (4):</u> Graph G has a back edge of case C and (up to reversal) every back edge of G is a back edge of either case C, B, A or E.

<u>m = 2:</u>

We form cases based on j_2.

Case Cα: $j_2 < j_1$. ($j_2 = a_1$, or not $a_2 < i_2 < b_2$)

Case Cβ: $j_1 \leq j_2 < b_1$. (not $a_2 < i_2 < b_2$)

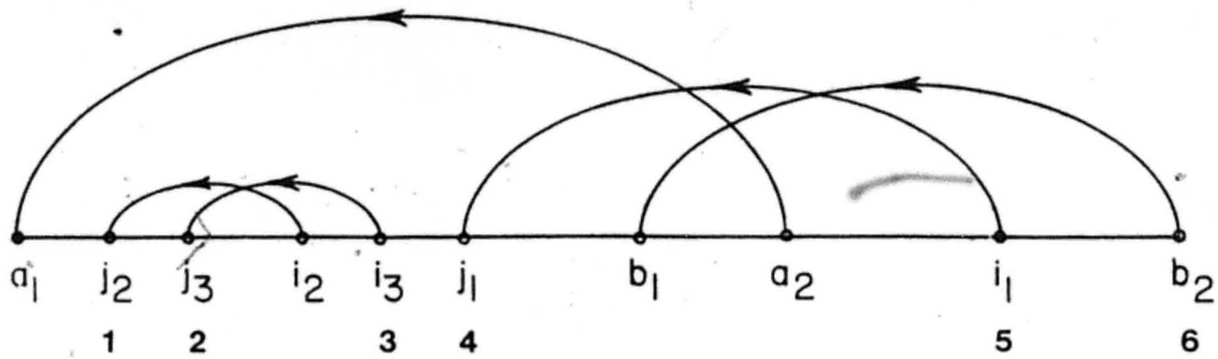


Figure 5.50

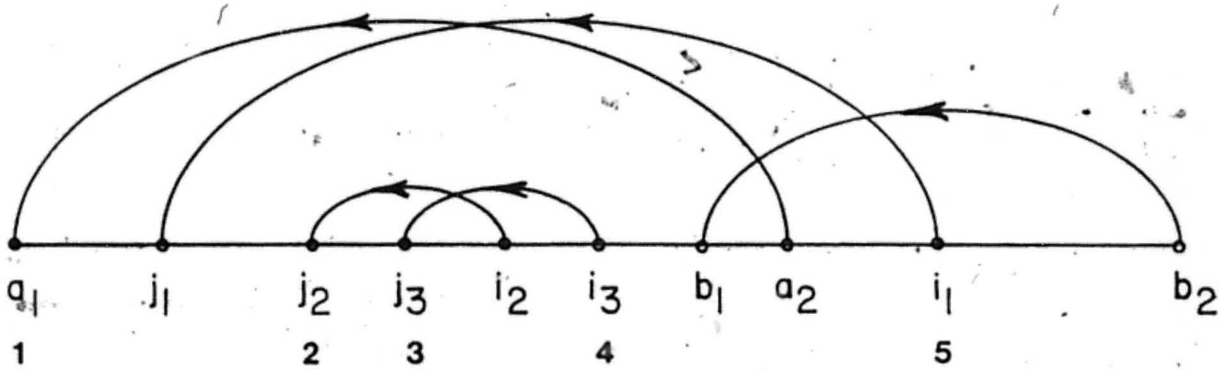


Figure 5.51

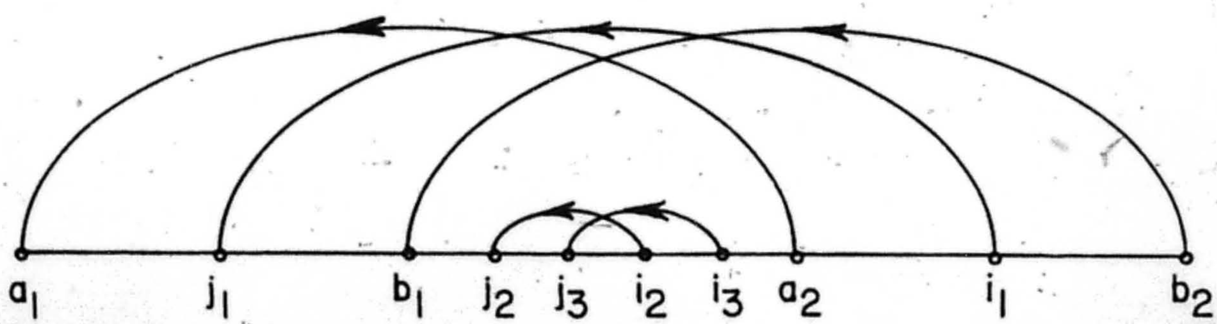


Figure 5.52

Case Cγ: $b_1 \leq j_2 \leq i_1$.

Case Cδ: $i_1 < j_2 \leq a_2$.

Case Cε: $a_2 < j_2$. In this case the edge i_2j_2 is useless, a contradiction. (See Figure 5.53)

We form subcases of case Cα based on i_2.

Case Cα1: $i_2 < j_1$. In this case the edge i_2j_2 is useless, a contradiction. (See Figure 5.54)

Case Cα2: $j_1 \leq i_2 < b_1$. Here make three further divisions:

(i) $j_2 > a_1$. In this case, G mimics MIN.16. (See Figure 5.55)

(ii) $j_2 = a_1$, $i_1 < a_2$. In this case, G mimics MIN.34. (See Figure 5.56)

(iii) $i_1 = a_2$, $j_2 = a_1$. In this case, a reduction of G is mimicked by MAX.11. (See Figure 5.57)

Case Cα3: $b_1 \leq i_2 \leq i_1$ Here make two further divisions:

(i) $i_2 < i_1$. In this case, G mimics MIN.33. (See Figure 5.58)

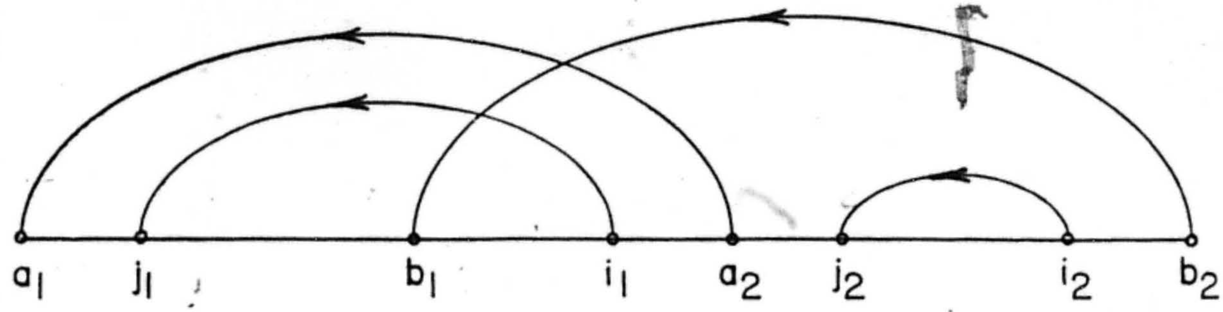


Figure 5.53

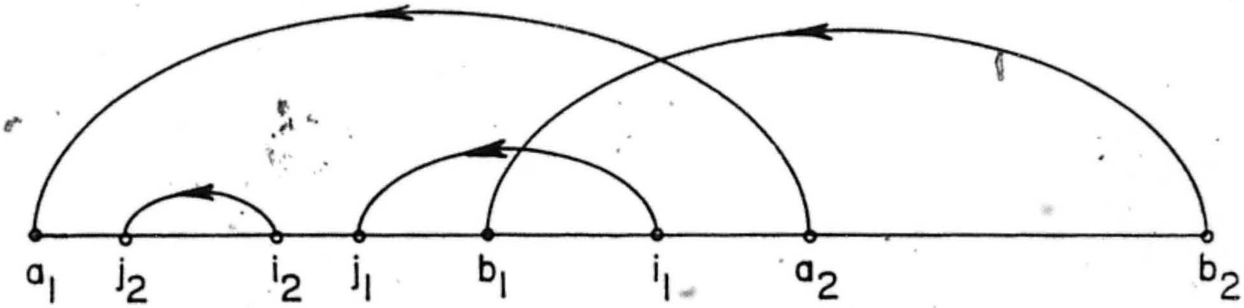


Figure 5.54

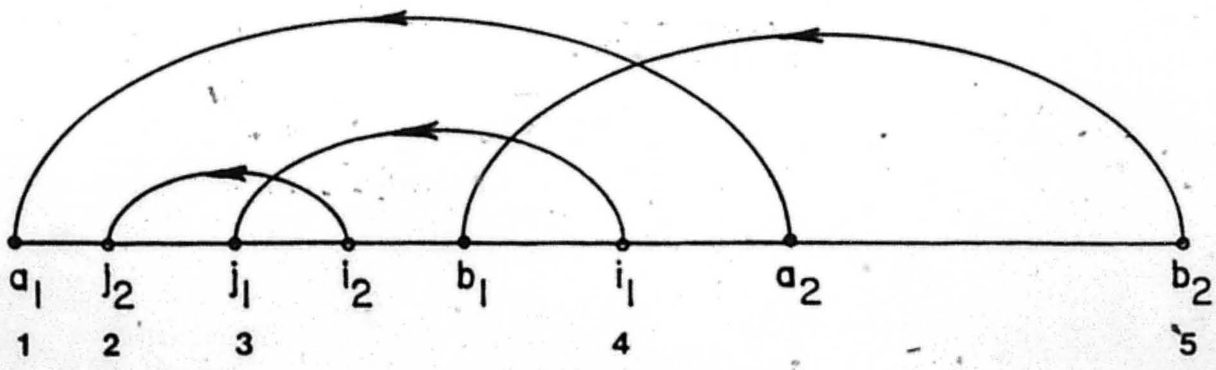


Figure 5.55

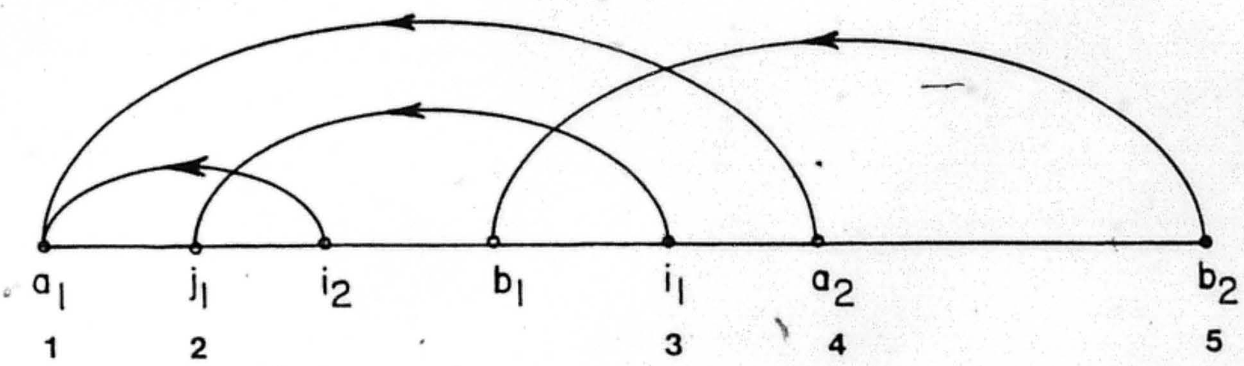


Figure 5.56

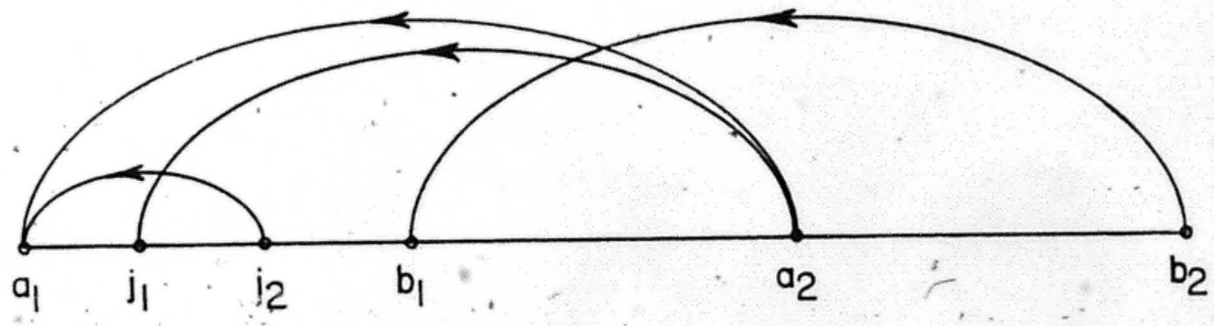


Figure 5.57

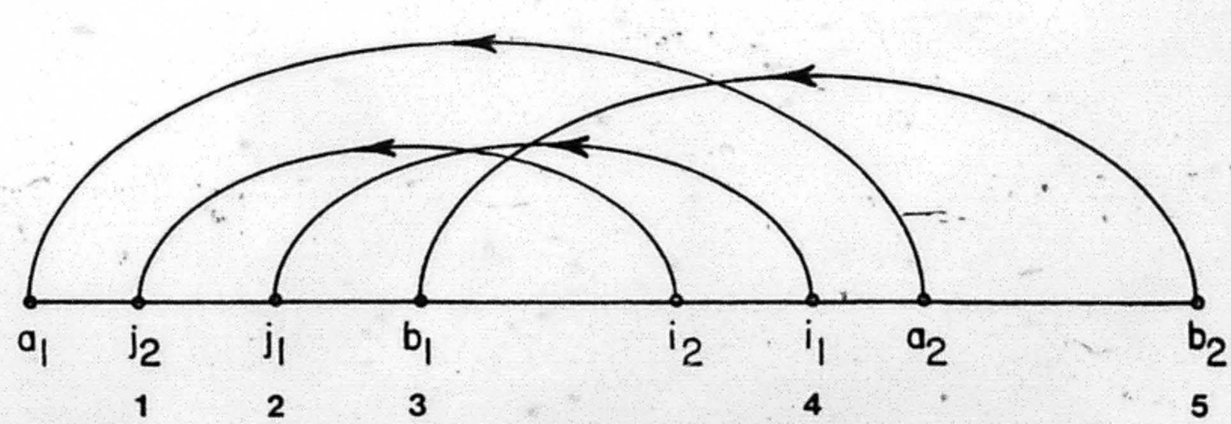


Figure 5.58

(ii) $i_2 = i_1$. In this case, a reduction of G is mimicked by MAX.10. (See Figure 5.59)

Case Cα4: $i_1 < i_2 \leq a_2$. Here make two further divisions:

(i) $a_1 = j_2$. In this case, G mimics MIN.36. (See Figure 5.60)

(ii) $a_1 < j_2$. In this case, G mimics MIN.37. (See Figure 5.61)

Case Cα5: $a_2 < i_2 < b_2$. Here $a_1 = j_2$, or else G has a type D edge. Interchanging the roles of $a_2 a_1$ and $i_2 j_2$ gives case Cα4 or Cα3 (ii) when $i_1 = a_2$.
(See Figure 5.62)

Case Cα6: $i_2 = b_2$. In this case, a reduction of G is mimicked by MAX.9. (See Figure 5.63)

Case Cβ is subdivided as follows depending on i_2:

Case Cβ1: $i_2 < b_1$. In this case the edge $i_2 j_2$ is useless, a contradiction. (See Figure 5.64)

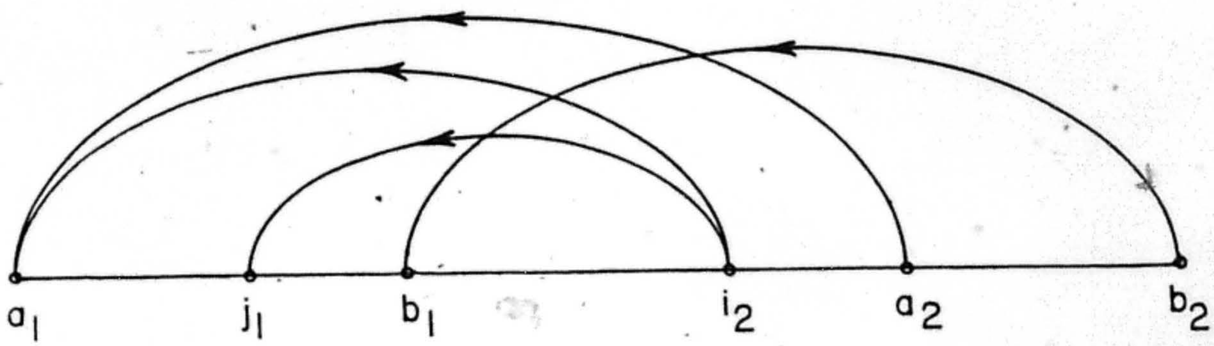


Figure 5.59

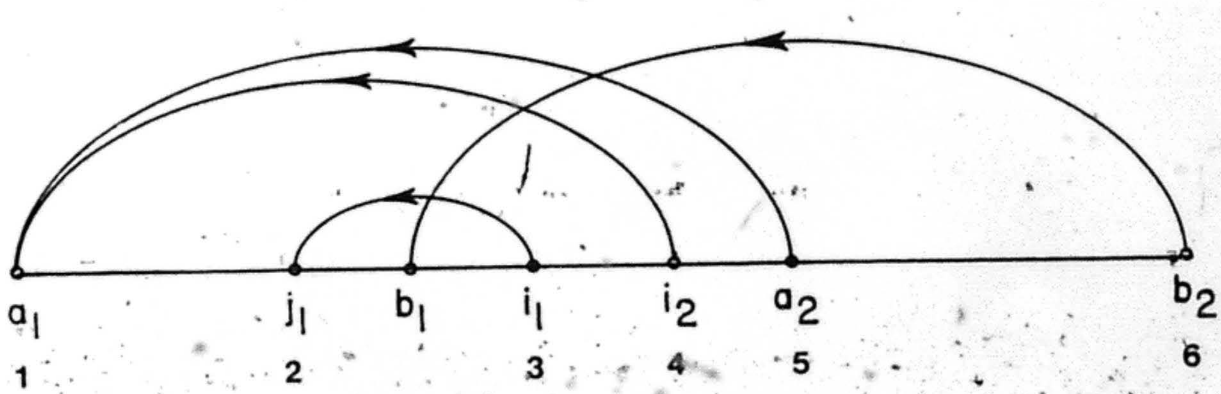


Figure 5.60

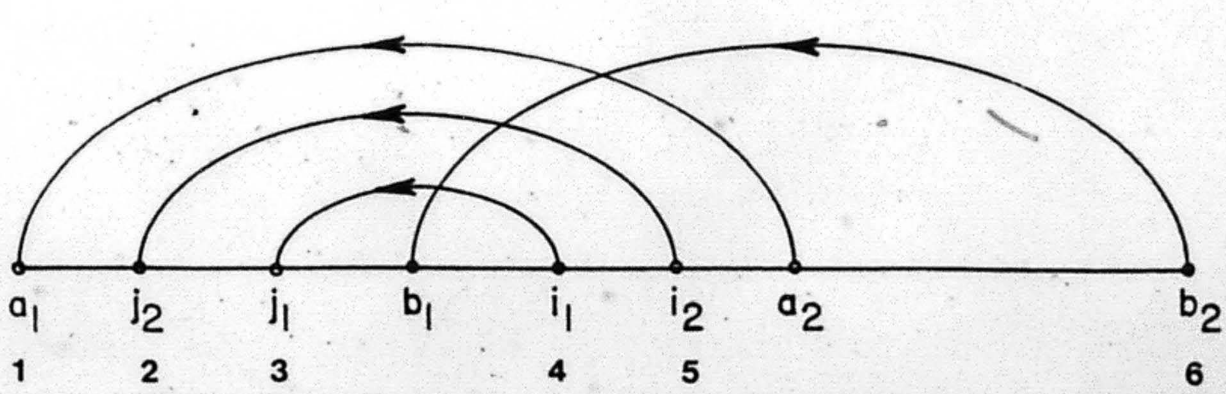


Figure 5.61

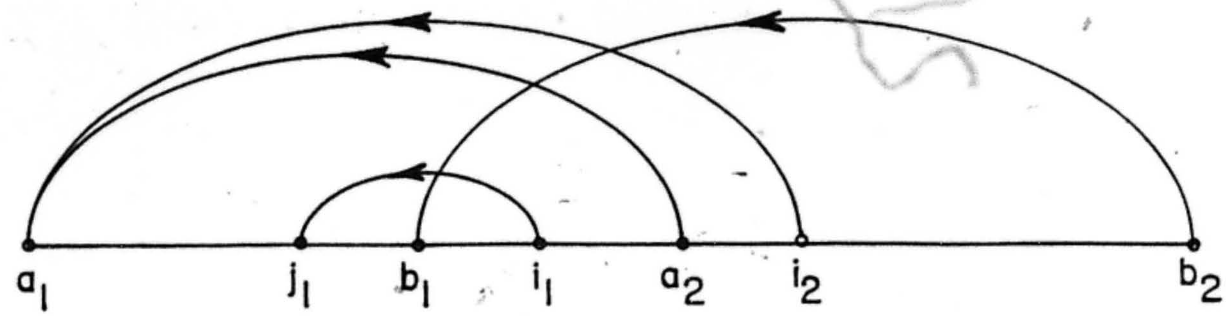


Figure 5.62

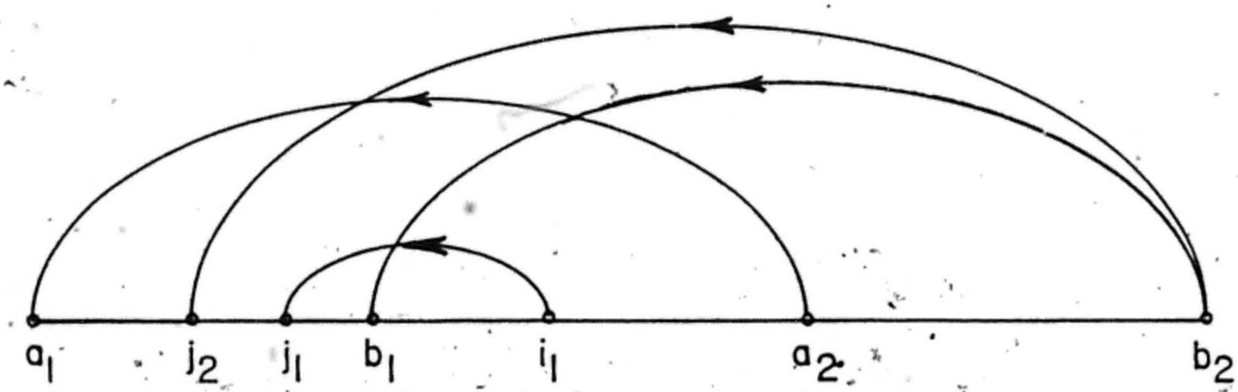


Figure 5.63

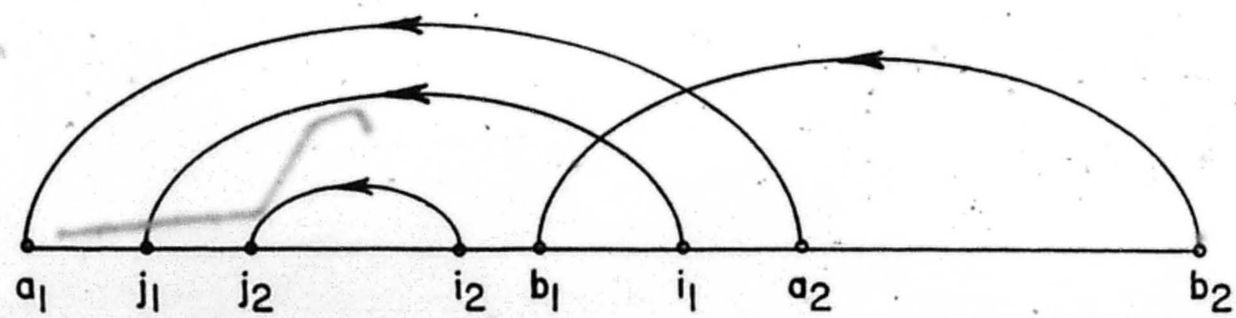


Figure 5.64

Case $C\beta 2$: $b_1 \leq i_2 \leq a_2$. Here we assume $j_1 = j_2$. Otherwise, reversing the roles of i_1j_1 and i_2j_2 gives rise to case $C\alpha 3$ or $C\alpha 4$. In this case, a reduction of G is mimicked by MAX.2. (See Figure 5.65)

Case $C\beta 3$: $a_2 < i_2$. (Thus assume $i_2 = b_2$, since G has no edges of type D.). Here make two further divisions:

(i) $j_1 = j_2$. In this case, a reduction of G is mimicked by MAX.4.
(See Figure 5.66)

(ii) $j_1 < j_2$. In this case, G mimics MIN.38. (See Figure 5.67)

Case $C\gamma$ is subdivided as follows depending on i_2:

Case $C\gamma 1$: $i_2 \leq i_1$. In this case the edge i_2j_2 is useless, a contradiction. (See Figure 5.68)

Case $C\gamma 2$: $i_1 < i_2 < b_2$. In this case, if $i_1 < a_2$ then G mimics MIN.55. (See Figure 5.69. Here 4 labels the lesser of a_2, i_2 w.r.t. P.) If $i_1 = a_2$, then G mimics MIN.77. (See Figure 5.70.)

Case $C\gamma 3$: $i_2 = b_2$. Here interchanging names

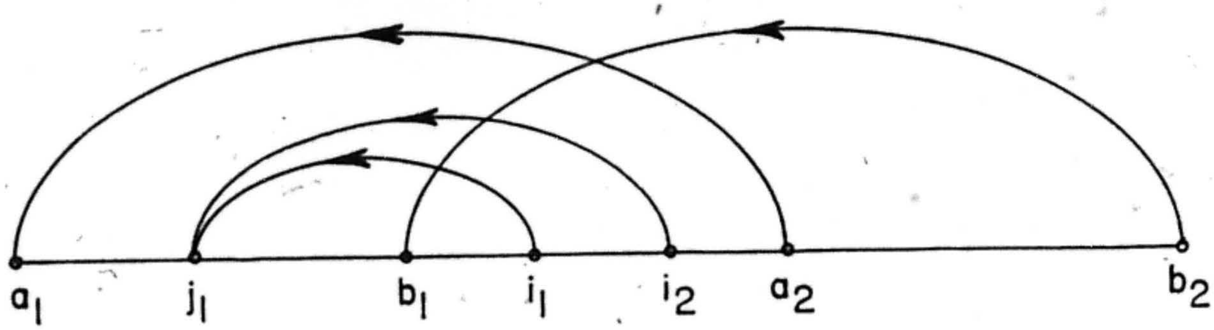


Figure 5.65

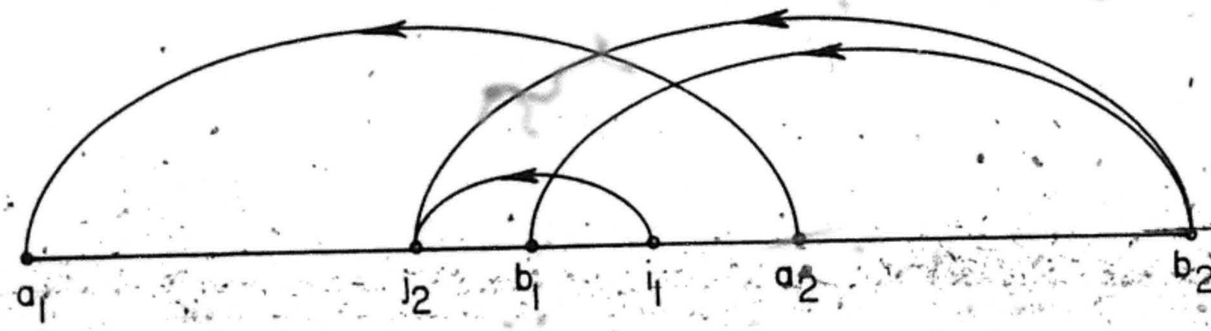
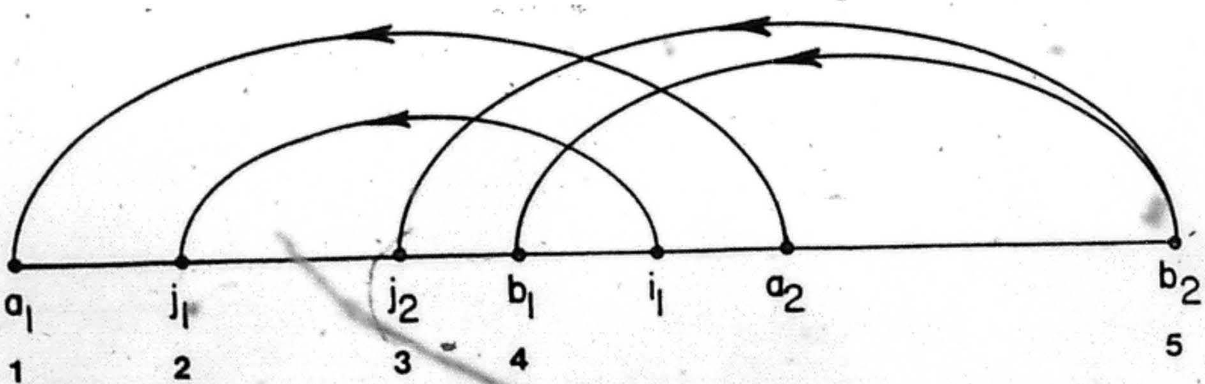


Figure 5.66



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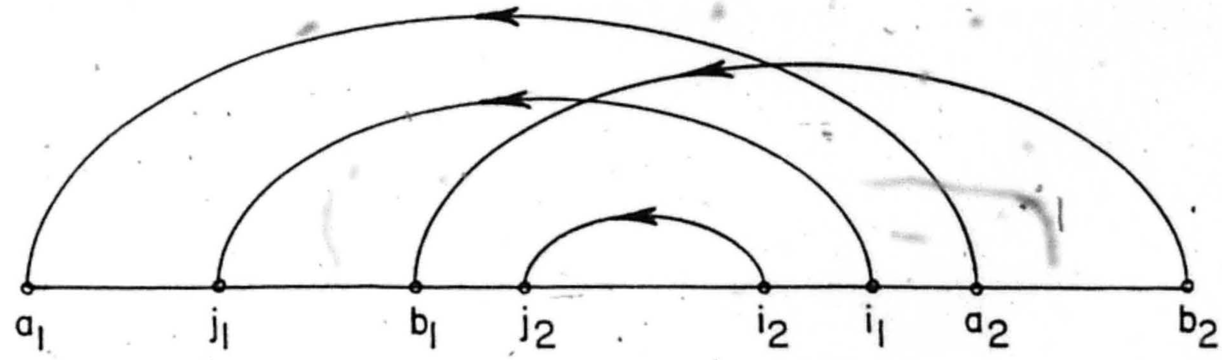


Figure 5.68

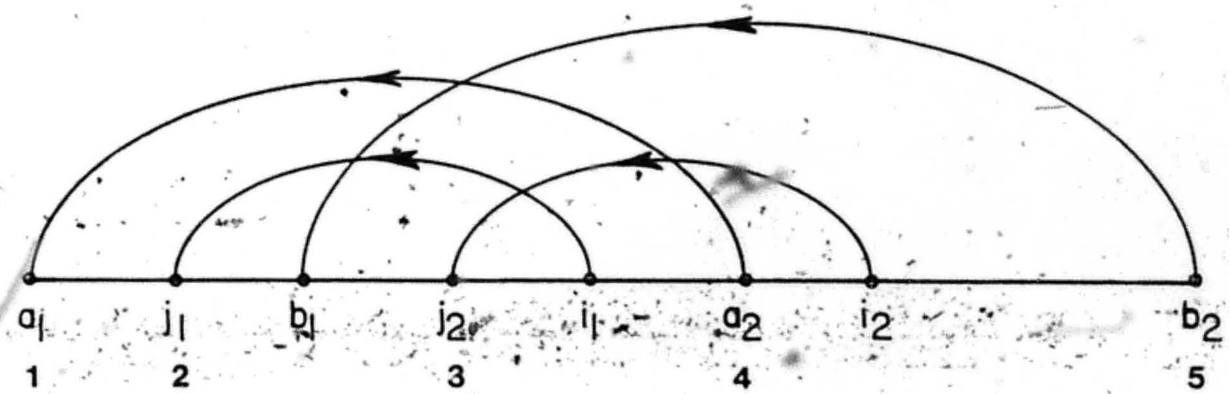


Figure 5.69

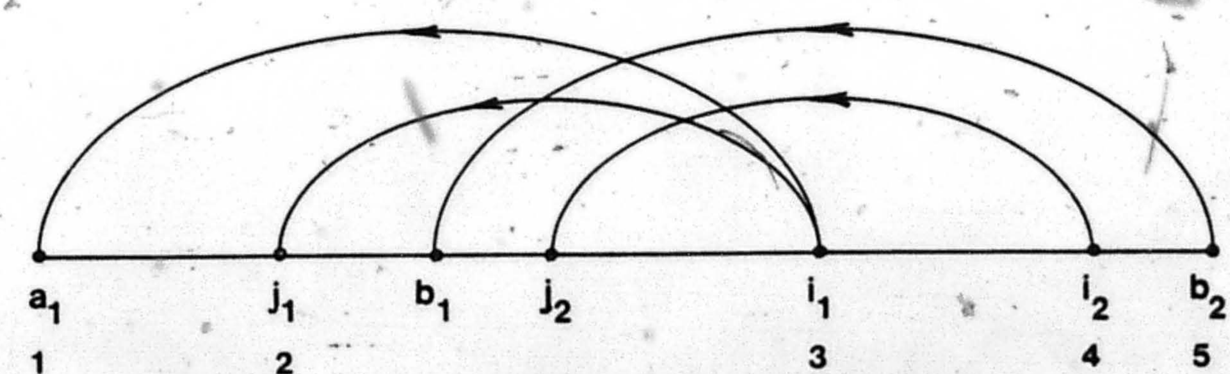


Figure 5.70

of i_2j_2 and b_2b_1 puts us in case Cβ3(ii) (See Figure 5.71.)

Case Cδ is subdivided as follows depending on i_2:

Case Cδ1: $i_2 \le a_2$. In this case the edge i_2j_2 is useless, a contradiction. (See Figure 5.72)

Case Cδ2: $a_2 < i_2 < b_2$. If G has no vertex between i_1 and j_2 on P, then a reduction of G is mimicked by MAX.8.

(See Figure 5.73)

However, if there is a vertex x of G between i_1 and j_2, then G mimics MIN.40. (See Figure 5.74.)

Case Cδ3 $i_2 = b_2$. In this case, a reduction of G is mimicked by MAX.5. (See Figure 5.75)

This concludes the subcase when m = 2. In most cases we showed that G mimicked a graph of MIN. In cases Cα1, Cβ1, Cγ1, Cδ1, Cε the edge i_2j_2 was useless. The other cases were:

Cα3 (ii) with $a_1 < j_2$, Cβ2

Cα6, Cβ3 (i)

Cδ2, Cδ3

Cα2 (iii)

Figure 5.71

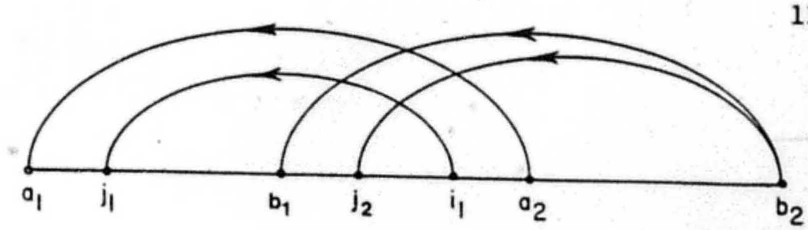


Figure 5.72

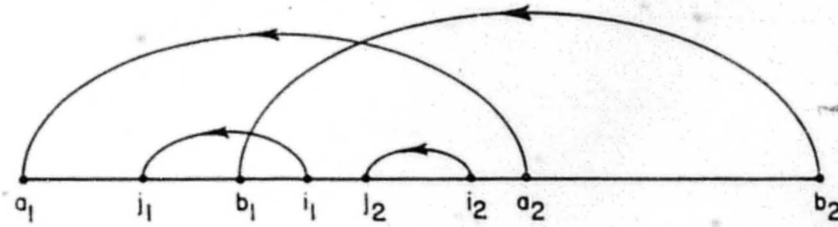


Figure 5.73

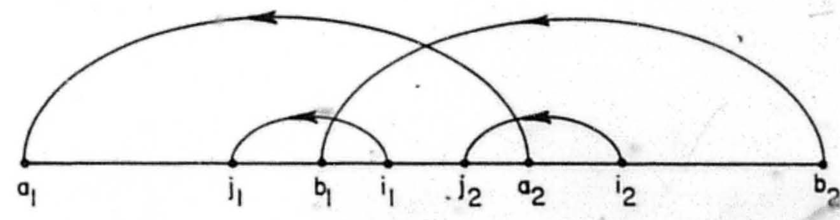


Figure 5.74

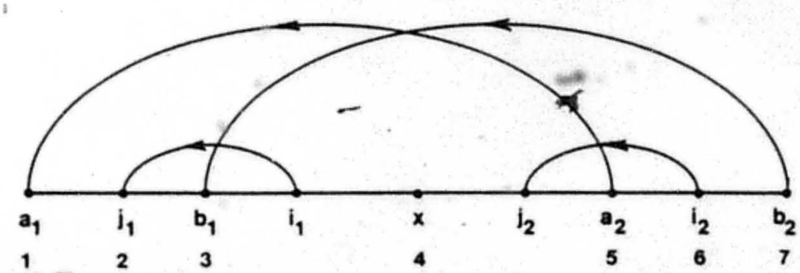
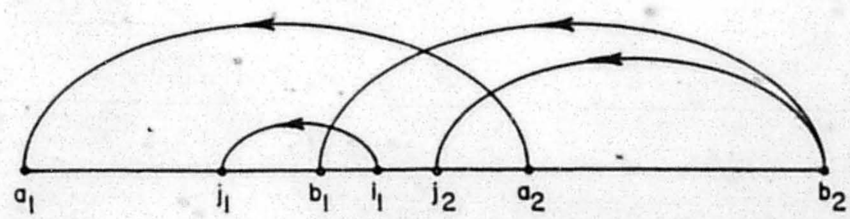


Figure 5.75



Cα3 (ii) with $a_1 = j_2$

We have grouped these cases according to similarities which are evident in the figures given for these cases. To aid the memory (not wishing the reader to have to recall what case Cα3 (ii) edges look like and so forth), we reflect these similarities in a renaming of cases; refer to edges falling under cases

Cα3 (iii), Cβ2 as type C1 (a), C1 (b) edges respectively,

Cα6, Cβ3 (i) as type C2 (a), C2 (b) edges respectively,

Cδ2, Cδ3 as type C3 (a), C3 (b) edges respectively,

Cα2 (iii) as type C4 edges,

Cα3 (ii) as type C5 edges.

Since these terms will be used in further breakdowns, the reader is advised to review the named cases so as to have at his finger tips what the edges of these various types look like.

Useless Edges:

Next we consider the cases when $m \geq 3$, and $i_2 j_2$ satisfies the conditions given in one of cases Cα1, Cβ1, Cγ1, Cδ1, Cε.

Edge i_2j_2 falls under Cα1: Without loss of generality, edge i_2j_2 forms an M with some edge i_3j_3 where $j_3 < b_1$. This gives three subcases:

Subcase- i_3j_3 is an edge of type C1(a): Without loss of generality, $a_1 \leq j_2 < j_3 \leq i_2 < j_1 < b_1 \leq i_3 = i_1$. In this case, G mimics MIN.39. (See Figure 5.76)

Subcase- i_3j_3 is an edge of type C2(a): Without loss of generality, $a_1 \leq j_2 < j_3 \leq i_2 < j_1 < i_3 = b_2$. In this case, G mimics MIN.41. (See Figure 5.77)

Subcase- i_3j_3 is an edge of type Cα1: Without loss of generality, $a_1 \leq j_2 < j_3 \leq i_2 < i_3 < j_1$. We make a further subdivision:

(i) $a_1 < j_2$. In this case, G mimics MIN.18. (See Figure 5.78)

(ii) $a_1 = j_2$, $i_1 < a_2$. In this case, G mimics MIN.27. (See Figure 5.79)

(iii) $a_1 = j_2$, $i_1 = a_2$ and $m = 3$. In this case, a reduction of G is mimicked by MAX.7. (See Figure 5.80)

Figure 5.76

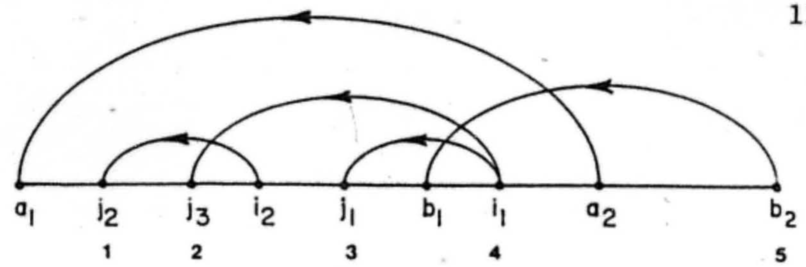


Figure 5.77

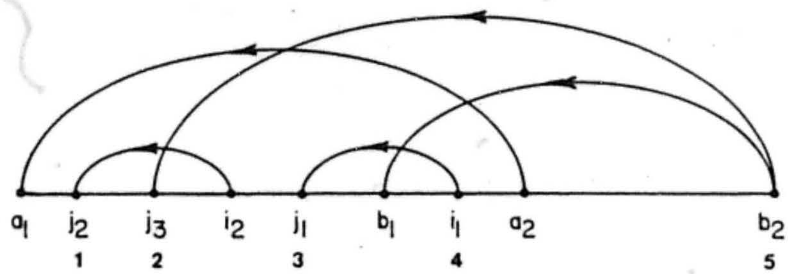


Figure 5.78

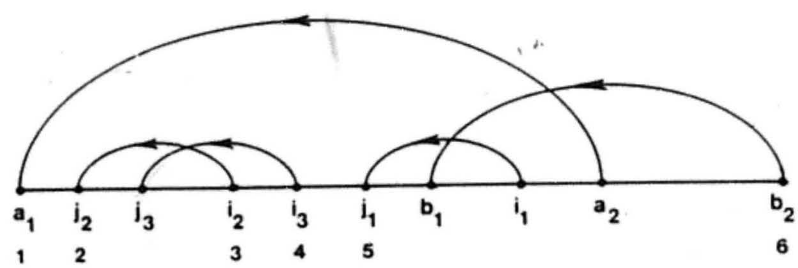


Figure 5.79

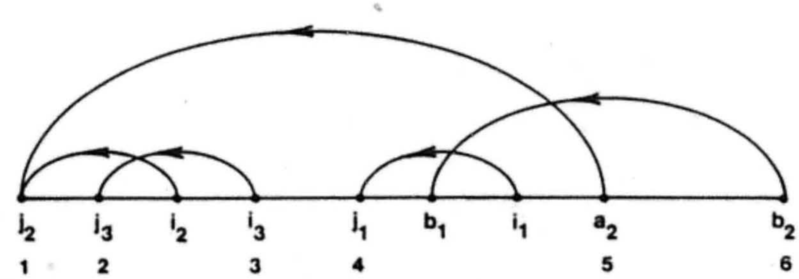
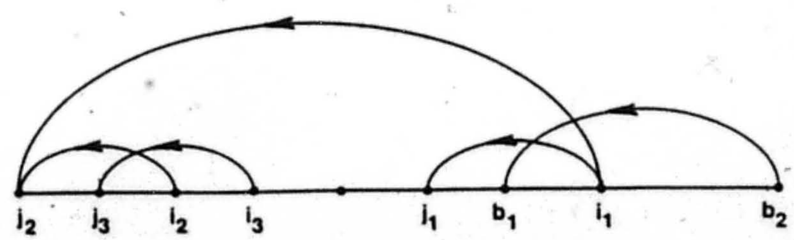


Figure 5.80



What happens if i_2j_2 and i_3j_3 are Cα1 edges forming an M, $a_1 = j_2$, $i_1 = a_2$ and $m > 3$? We shall see in a moment that if $m \geq 3$ and G has an edge of type Cβ1, Cγ1, Cδ1, or Cε, then G mimics a graph of MIN. We shall thus assume here that G has no such edges. We thus assume that edge i_4j_4 falls under one of cases Cα1, C1, C2, C3, C4, C5.

Since $i_1 = a_2$, it follows that G has no edges of types C1(b), C3 or C5. It also follows from previous discussion concerning the edges of type Cα1 that we may assume that the only edges forming M's with Cα1 edges are themselves Cα1 edges. Only a few cases are left:

G has an edge of type C1(a): Here G mimics MIN.27. (See Figure 5.81.)

G has an edge of type C2: Here G mimics MIN.27. (See Figure 5.82.)

G has an edge of type C4: Here G mimics MIN.14. (See Figure 5.83.)

Every extraskeletal edge of G other than i_1j_1 is a Cα1 edge: By our previous breakdown of the case when every extraskeletal edge of G is a type A edge, we may assume that G has an edge i_4j_4 where $a_1 = j_4 = j_2 < j_3 \leq i_2 < i_3 < i_4 < j_1 < b_1 \leq i_1 = a_2$. In

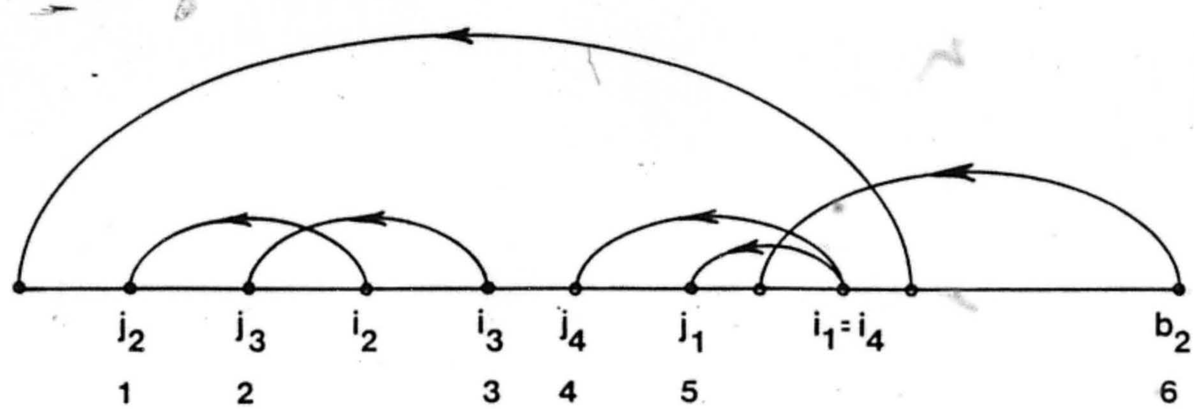


Figure 5.81

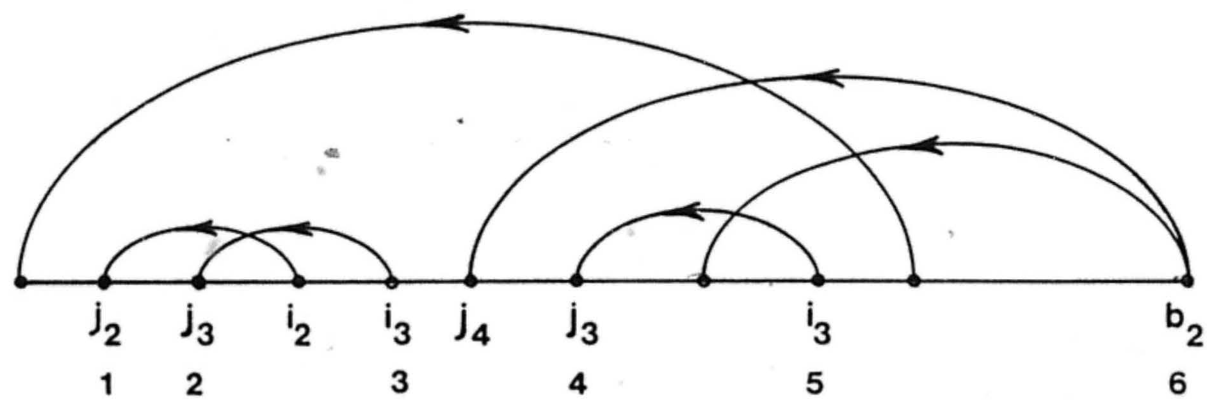


Figure 5.82

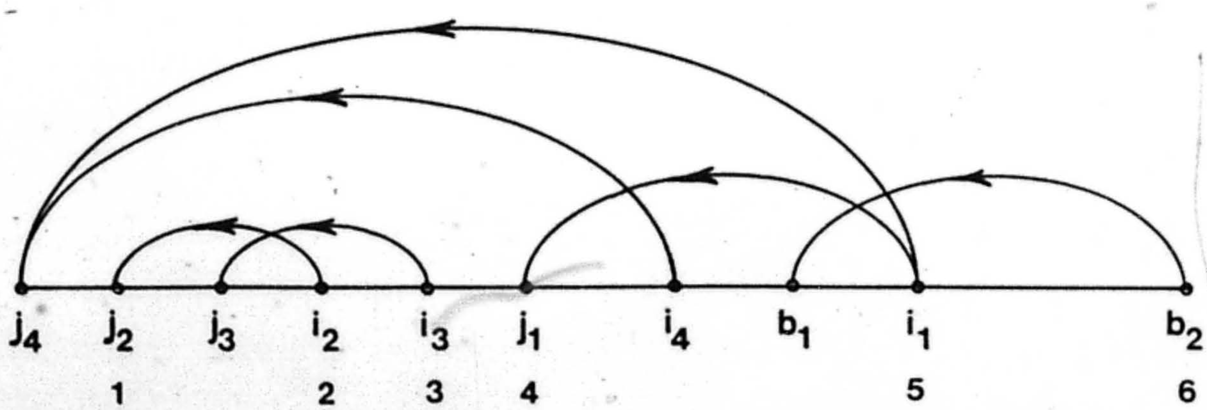


Figure 5.83

this case, G mimics MIN.42. (See Figure 5.84.)

This concludes the case where G has an edge of type $C\alpha 1$.

<u>Edge i_2j_2 falls under $C\beta 1$:</u> Without loss of generality, edge i_2j_2 forms an M with some edge i_3j_3 where $j_1 < j_3 < b_1$ and/or $j_1 \leq i_3 < b_1$. The candidate subcases are:

Subcase- i_3j_3 is an edge of type $C\beta 1$: Without loss of generality, $j_1 \leq j_2 < j_3 \leq i_2 < i_3 < b_1$. In this case, G mimics MIN.15. (See Figure 5.85)

Subcase- i_3j_3 is an edge of type C1(a). (Recall that in case $C\alpha 3$ we assumed that $j_2 < j_1$, and dismissed most of case $C\beta 2$ (where $j_1 < j_2$) as being the same case under renaming. However, with the presence of an additional edge, we must allow the possibility of $j_1 < j_2$ as separate.): Without loss of generality, $j_1 \leq j_2 < j_3 \leq i_2 < i_3 = i_1$. In this case, G mimics MIN.16. (See Figure 5.86)

Subcase- i_3j_3 is an edge of type C4: Without loss of generality, $a_1 = j_3 < j_1 \leq j_2 \leq i_3 < i_2$. In this case, G mimics MIN.43. (See Figure 5.87)

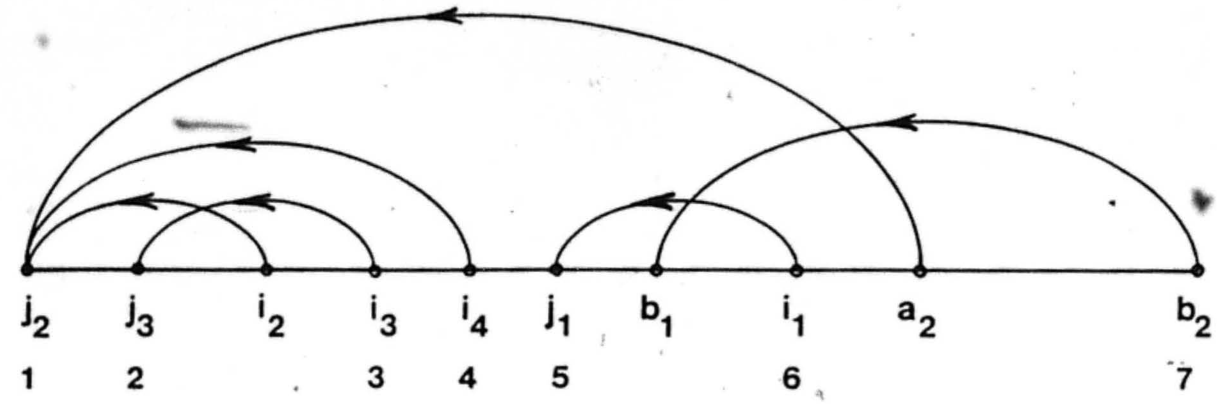


Figure 5.84

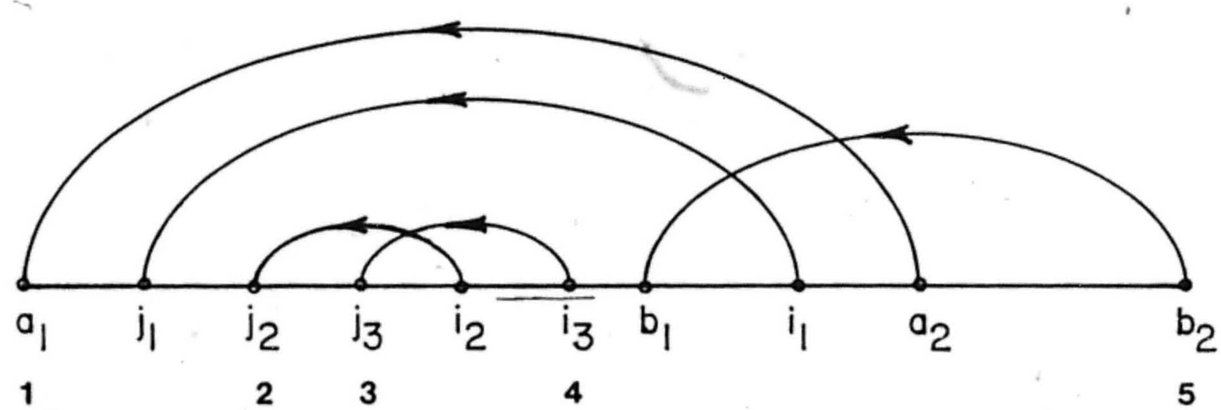


Figure 5.85

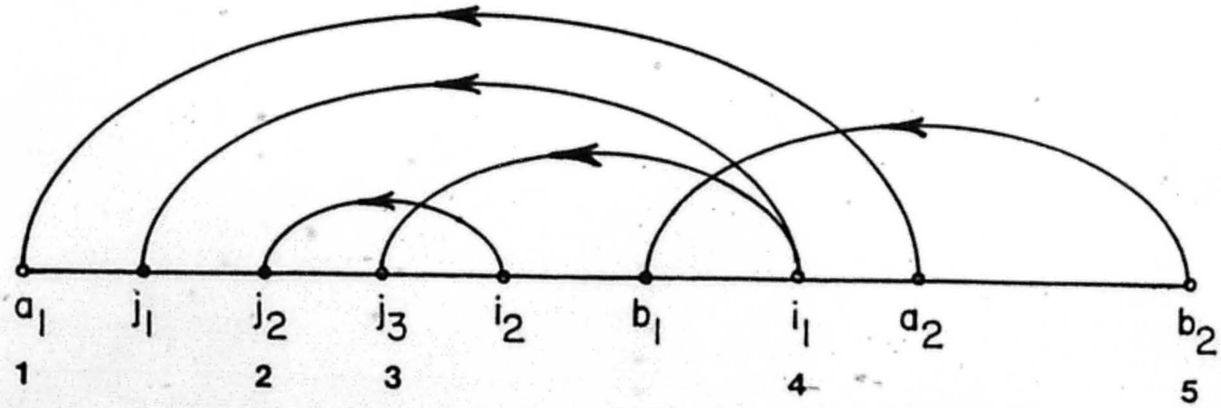


Figure 5.86

<u>**Edge i_2j_2 falls under Cγ1:**</u> Without loss of generality, edge i_2j_2 forms an M with some edge i_3j_3 where $b_1 < j_3 \leq i_1$ and/or $b_1 \leq i_3 < i_1$. The candidate subcases are:

Subcase- i_3j_3 is an edge of type Cγ1: In this case G has two type E edges forming an M, and we are done as in case (1).

Subcase- i_3j_3 is an edge of type C1(b). Without loss of generality, $j_1 = j_3 \leq b_1 \leq j_2 \leq i_3 < i_2 \leq i_1$. In this case, G mimics MIN.55. (See Figure 5.88)

<u>**Edge i_2j_2 falls under Cδ1:**</u> Without loss of generality, edge i_2j_2 forms an M with some edge i_3j_3 where $i_1 < j_3 \leq a_2$ and/or $i_1 < i_3 < a_2$. The candidate subcases are:

Subcase- i_3j_3 is an edge of type Cδ1: In this case G has two type E edges forming an M, and we are done as in case (1).

Subcase- i_3j_3 is an edge of type C1(b). Without loss of generality, $j_1 = j_3 < b_1 \leq i_1 < j_2 \leq i_3 < i_2 \leq a_2$. In this case, G mimics MIN.15. (See Figure 5.89)

Subcase- i_3j_3 is an edge of type C3. Actually

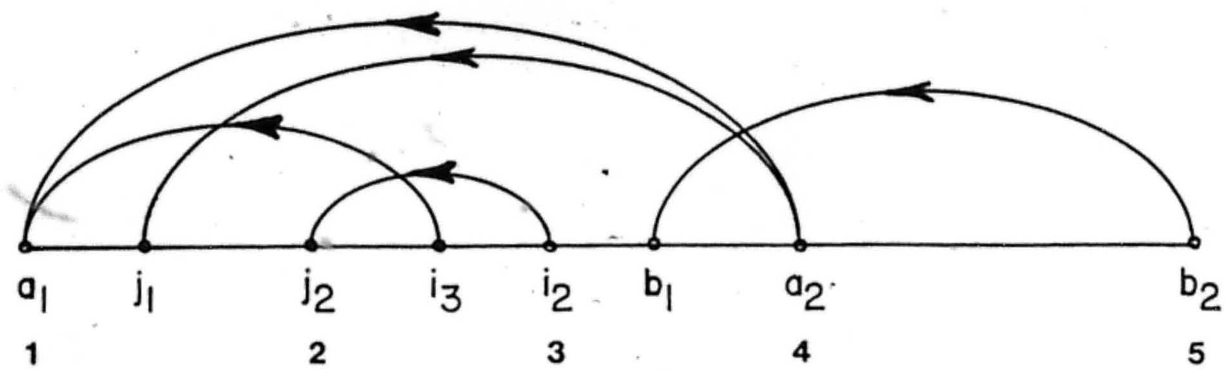


Figure 5.87

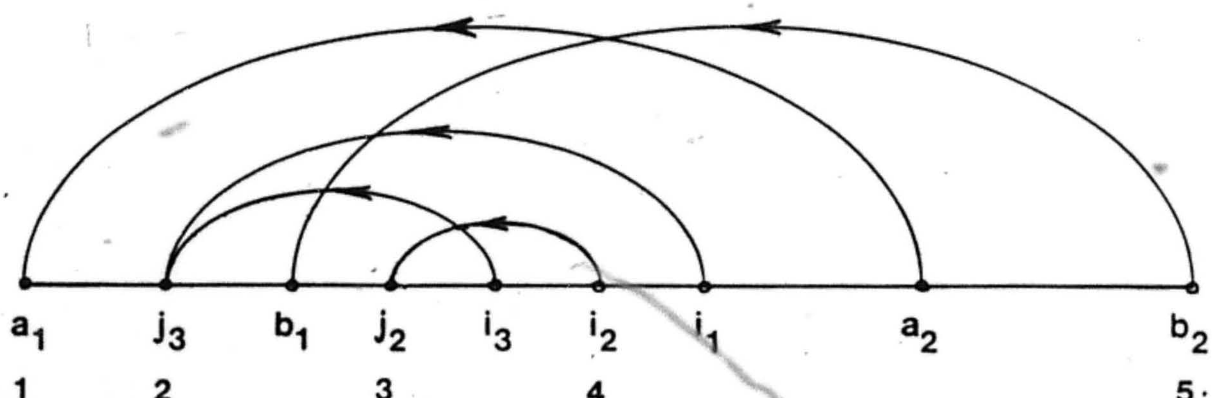


Figure 5.88

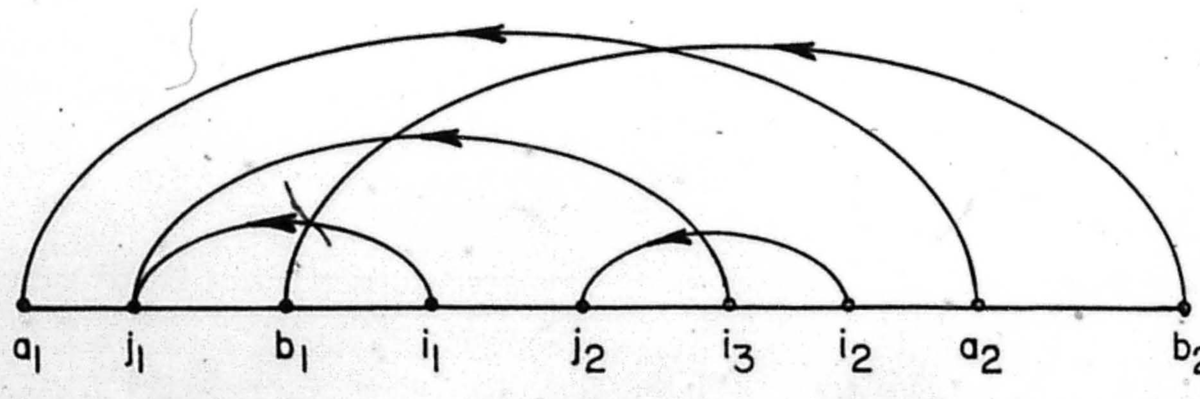


Figure 5.89

in C3(a), there is no vertex between i_1 and j_2 (= j_3 here) so only C3(b) is possible. Without loss of generality, $b_1 \leq i_1 < j_2 < j_3 \leq i_2 \leq a_2 < i_3$. In this case, G mimics MIN.41. (See Figure 5.90)

Edge $i_2 j_2$ falls under $C\epsilon$: Without loss of generality, edge $i_2 j_2$ forms an M with some edge $i_3 j_3$ where $a_2 < j_3$ and/or $a_2 < i_3$. The candidate subcases are:

Subcase- $i_3 j_3$ is an edge of type $C\epsilon$. Without loss of generality, $a_2 < j_3 < j_2 \leq i_3 < i_2$. In this case, G mimics MIN.56.
(See Figure 5.91)

Subcase- $i_3 j_3$ is an edge of type C3(a). Without loss of generality, $i_1 < j_3 \leq a_2 < j_2 \leq i_3 < i_2$. In this case, G mimics MIN.34. (See Figure 5.92)

This completes the cases where $m \geq 3$, and $i_2 j_2$ satisfies the conditions given in one of cases $C\alpha 1$, $C\beta 1$, $C\gamma 1$, $C\delta 1$, $C\epsilon$.

No Useless Edges

From now on we assume that G does not have edges of types $C\alpha 1$, $C\beta 1$, $C\gamma 1$, $C\delta 1$, $C\epsilon$. Thus every extraskeletal edge of G is of one of types C1, C2, C3, C4, C5.

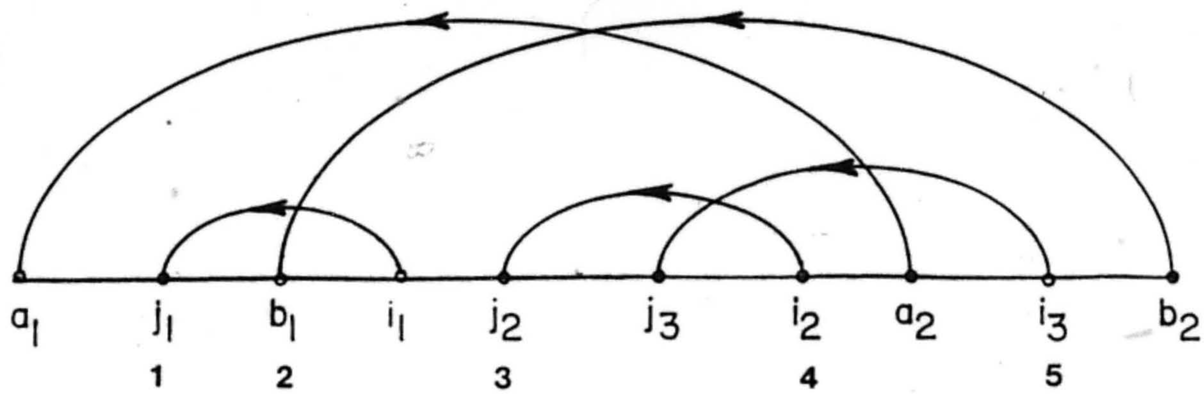


Figure 5.90

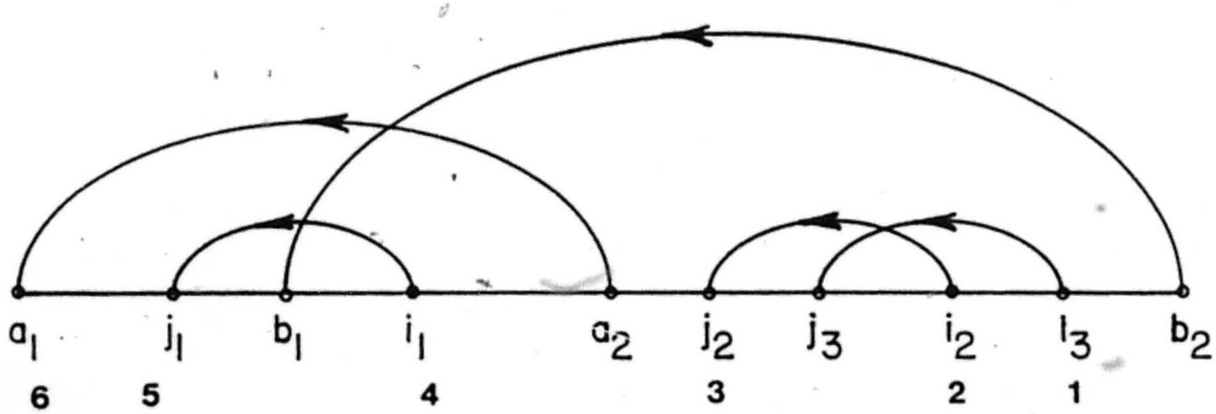


Figure 5.91

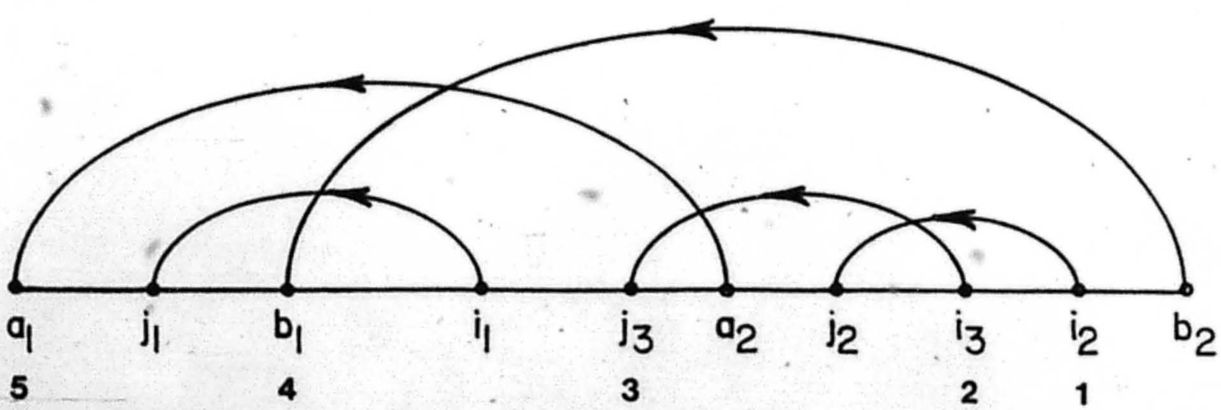


Figure 5.92

<u>m = 3:</u>

Another level of cases, will prove useful. We create five new cases:

(5.1) G contains a C5 edge i_2j_2.

(5.2) G contains a C4 edge i_2j_2, but no C5 edge.

(5.3) G contains a C3 edge i_2j_2, but no C4 or C5 edges.

(5.4) G contains a C2 edge i_2j_2, but no C3, C4 or C5 edges.

(5.5) G contains only C1 edges.

Case (5.1) is subdivided according to i_3j_3: Note that i_3j_3 cannot be a C4 edge, since $i_1 \neq a_2$. Also i_3j_3 cannot be a C5 edge, since then i_2j_2 and i_3j_3 would be equal.

<u>Subcase C5 + C3(a):</u> Edge i_3j_3 is a C3(a) edge. In this case, G mimics MIN.44. (See Figure 5.93)

<u>Subcase C5 + C3(b):</u> Edge i_3j_3 is a C3(b) edge. We have two possibilities:

(i) There is a vertex x of P between i_1 and j_3. In this case, G mimics MIN.45. (See Figure 5.94)

(ii) There is no vertex of P between i_1 and j_3. In this case, a reduction of G is mimicked by MAX.6. (See Figure 5.95)

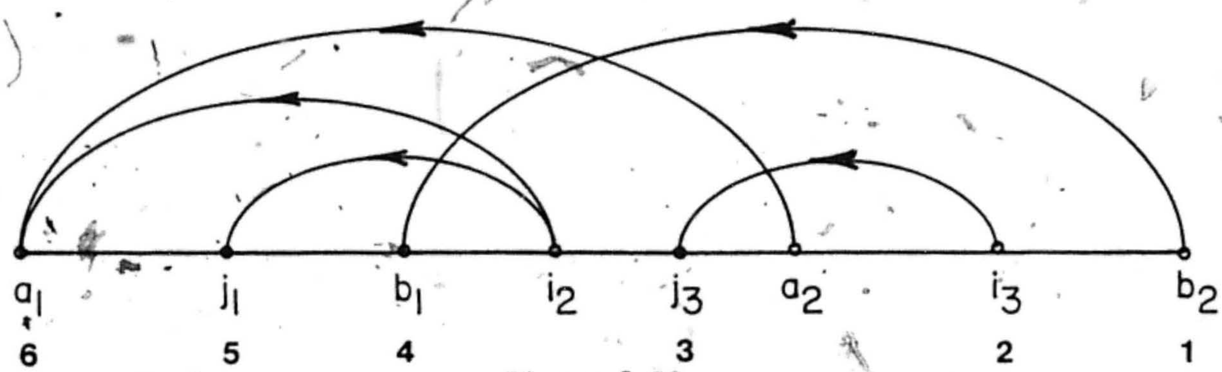


Figure 5.93

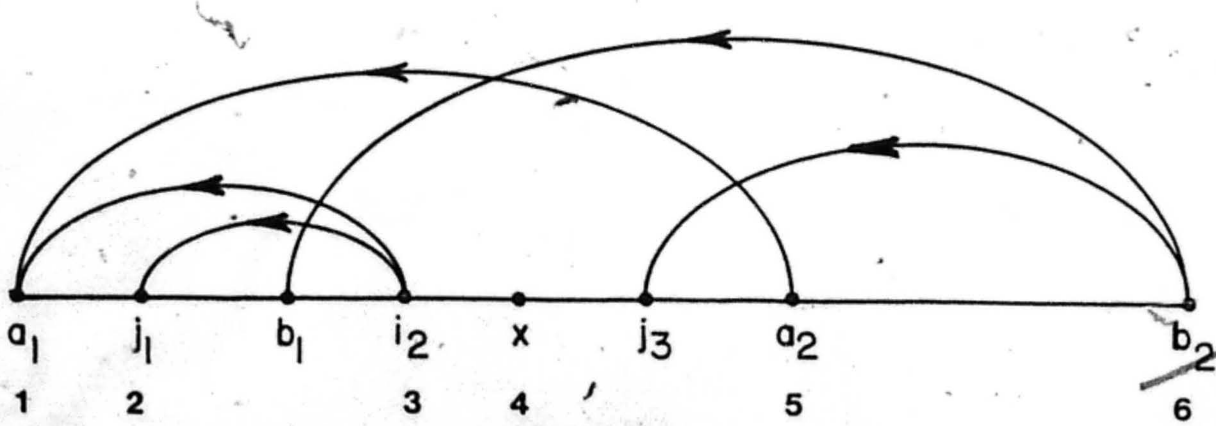


Figure 5.94

Subcase C5 + C2: Edge i_3j_3 is a C2 edge. In this case, G mimics MIN.28. (See Figure 5.96)

Subcase C5 + C1: Edge i_3j_3 is a C1 edge. Then i_3j_3 may be assumed to be a C1(a) edge, or else by considering edges i_3j_3 and i_2j_2, G falls under one of case Cα3(i), or case Cα4.

In this case, G mimics MIN.46. (See Figure 5.97)

Case (5.2) is subdivided according to i_3j_3:

Note that i_3j_3 cannot be a C3 edge, since $i_1 = a_2$.

Subcase C4 + C4: Edge i_3j_3 is a C4 edge. In this case, G mimics MIN.5. (See Figure 5.98)

Subcase C4 + C2: Edge i_3j_3 is a C2 edge. In this case, G mimics MIN.47. (See Figure 5.99)

Subcase C4 + C1(a): Edge i_3j_3 is a C1(a) edge. In case C1(a) (nee Cα3) we assumed without loss of generality that $j_2 < j_1$. With an additional C4 edge and relabelling this assumption gives way to two possibilities:

(i) $j_1 \le i_2 < j_3$. In this case, G mimics

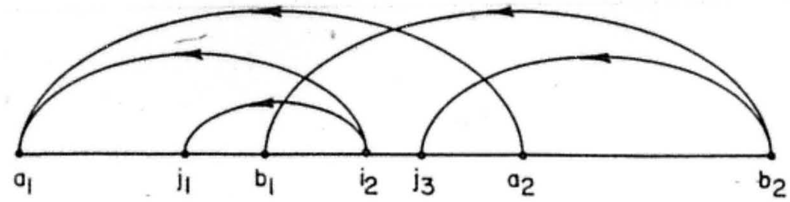


Figure 5.95

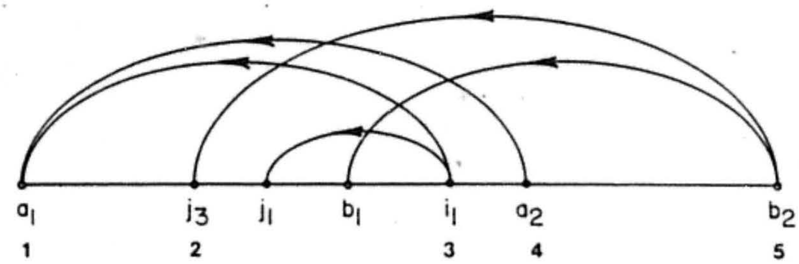


Figure 5.96

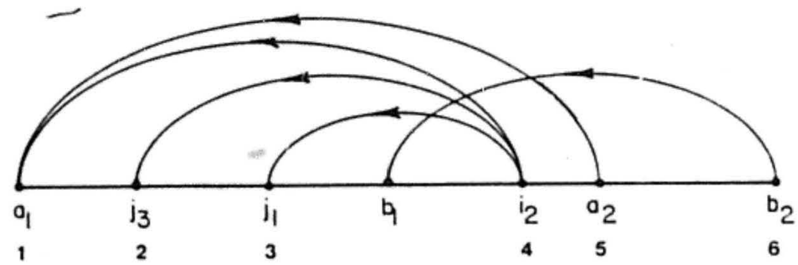


Figure 5.97

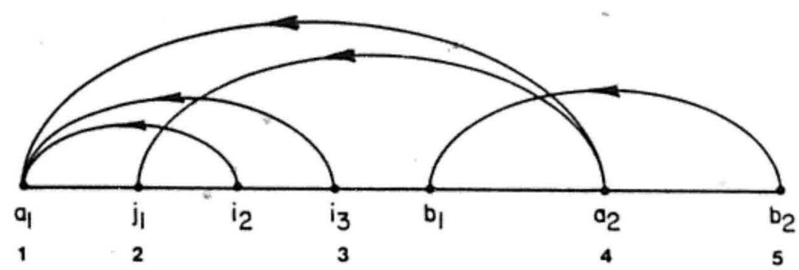


Figure 5.98

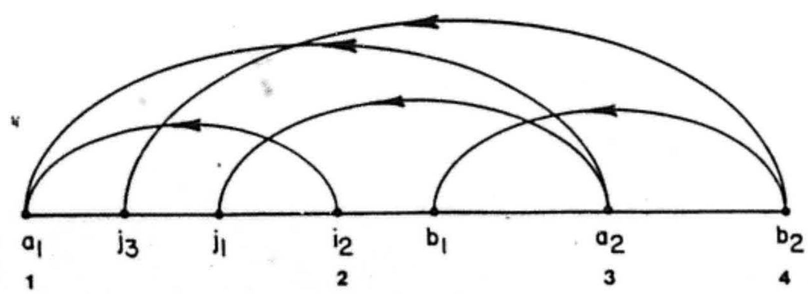


Figure 5.99

MIN.39.

(See Figure 5.100)

(ii) $j_3 \leq i_2$. In this case, G mimics MIN.7.

(See Figure 5.101)

<u>Subcase C4 + C1(b):</u> Edge i_3j_3 is a C1(b) edge. In this case, G mimics MIN.7. (See Figure 5.102)

Case (5.3) is subdivided according to i_3j_3:

<u>Subcase C3 + C3:</u> Edge i_3j_3 is a C3 edge. We have two possibilities:

(i) At least one of i_2j_2, i_3j_3 is a C3(a) edge. In this case, G mimics MIN.44. (See Figure 5.103)

(ii) Both i_2j_2, i_3j_3 are C3(b) edges. In this case a reduction of G is mimicked by MAX.5. (See Figure 5.104)

<u>Subcase C3 + C2:</u> Edge i_3j_3 is a C2 edge. We have three possibilities:

(i) Edge i_3j_3 is a C2(a) edge. In this case G mimics MIN.48. (See Figure 5.105)

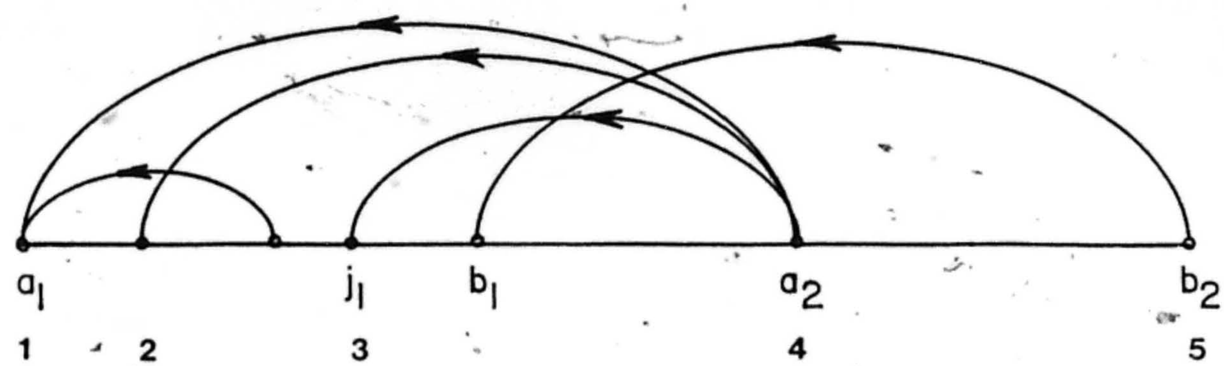


Figure 5.100

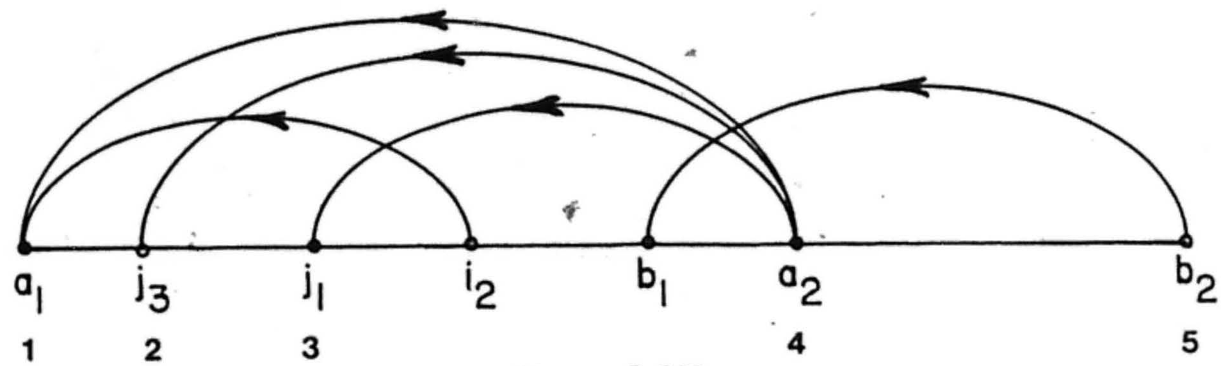


Figure 5.101

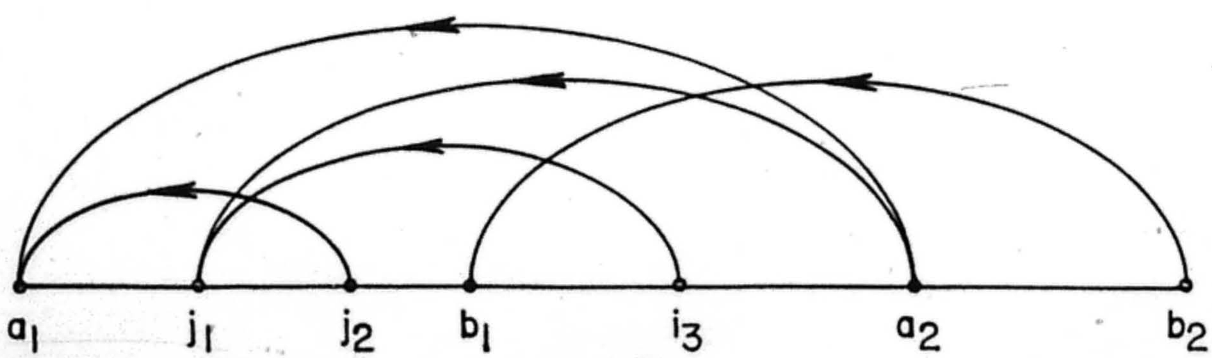


Figure 5.102

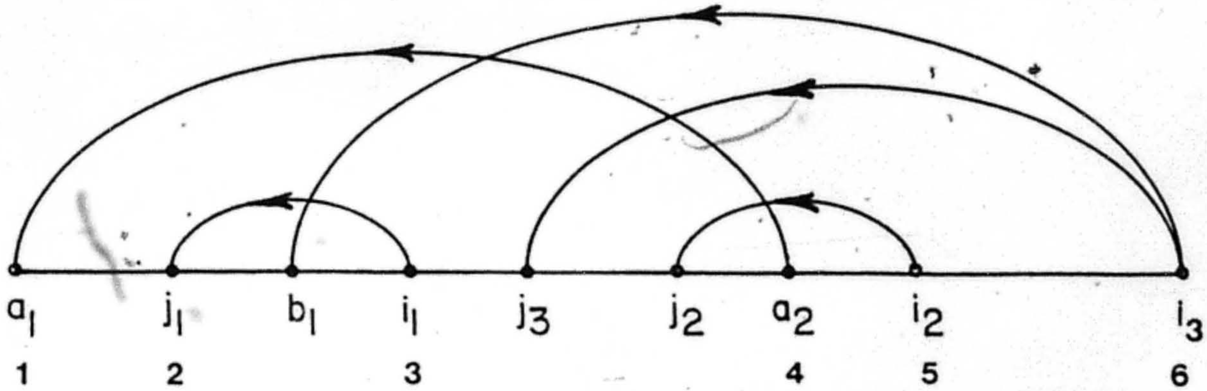


Figure 5.103

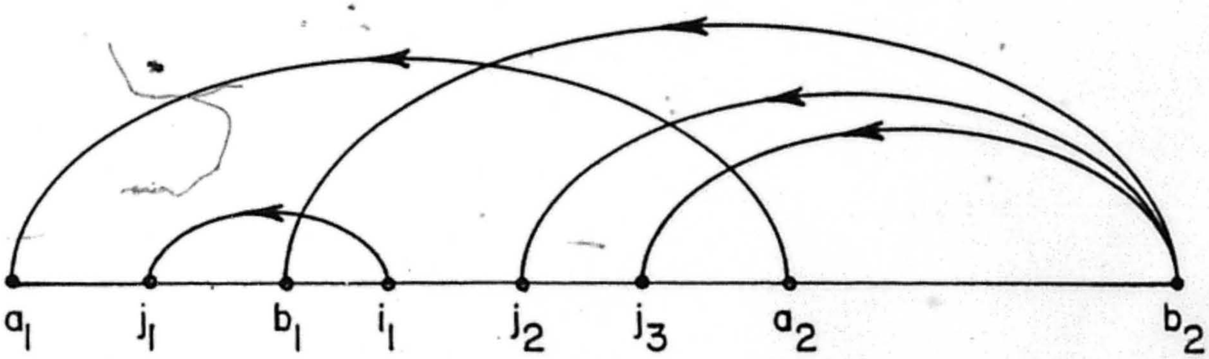


Figure 5.104

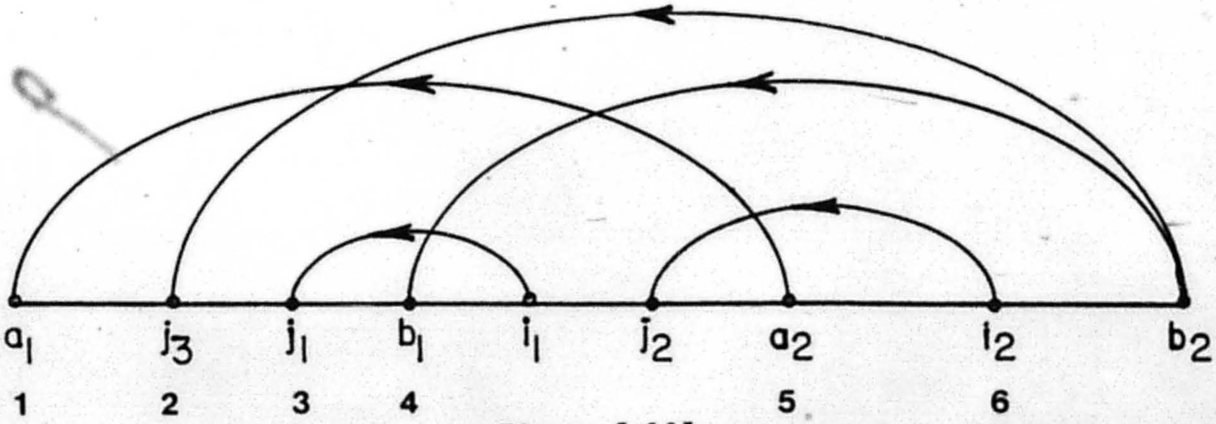


Figure 5.105

(i) Edge i_2j_2 is a C3(a) edge. In this case G mimics MIN.49. (See Figure 5.106)

(iii) Edge i_2j_2 is a C3(b) edge and edge i_3j_3 is a C2(b) edge. In this case a reduction of G is mimicked by MAX.4. (See Figure 5.107)

<u>Subcase C3 + C1(a):</u> Edge i_3j_3 is a C1(a) edge. In this case, G mimics MIN.50. (See Figure 5.108)

<u>Subcase C3 + C1(b):</u> Edge i_3j_3 is a C1(b) edge. In this case, G mimics MIN.51. (See Figure 5.109)

Case (5.4) is subdivided according to i_3j_3:

<u>Subcase C2 + C2:</u> Edge i_3j_3 is a C2 edge. In this case, G mimics MIN.52. (See Figure 5.110)

<u>Subcase C2 + C1:</u> Edge i_3j_3 is a C1 edge. We have three possibilities:

(i) Edge i_3j_3 is a C1(a) edge. Without loss of generality, $j_2 \leq j_3$, otherwise we get case Cβ3(ii). In this case G mimics MIN.53. (See Figure 5.111)

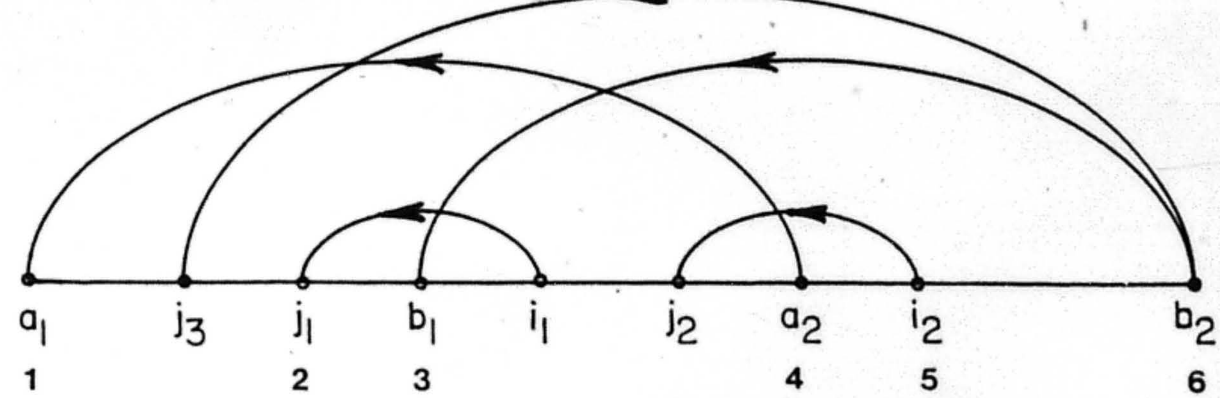


Figure 5.106

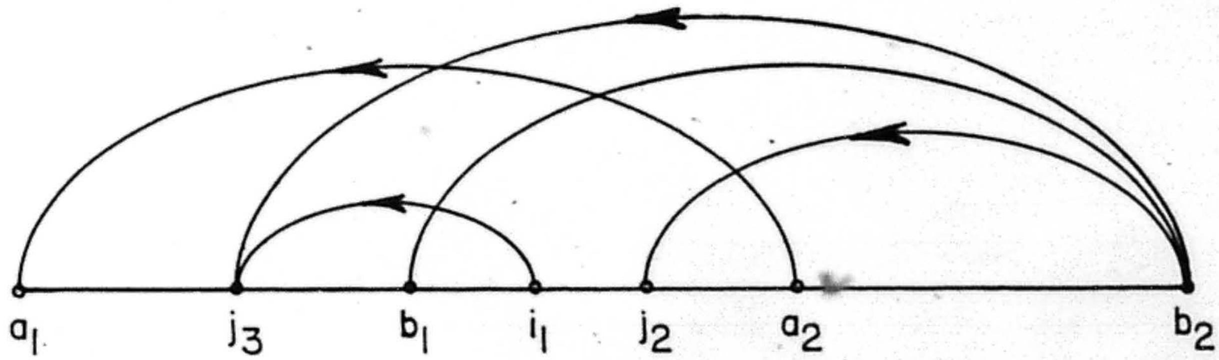


Figure 5.107

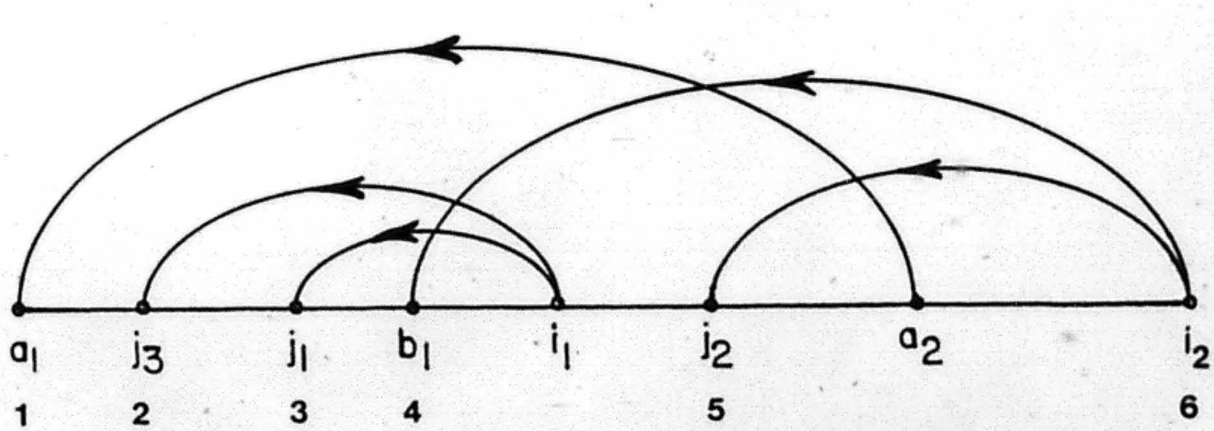


Figure 5.108

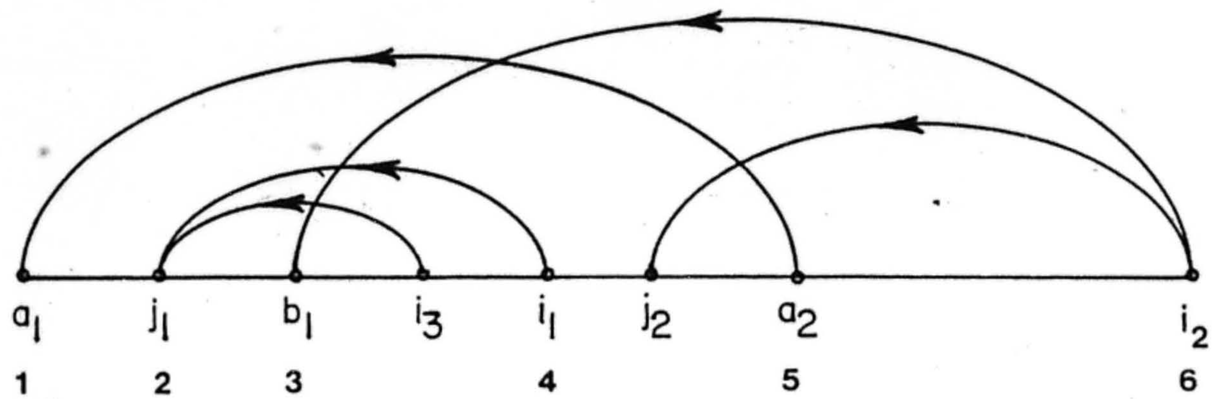


Figure 5.109

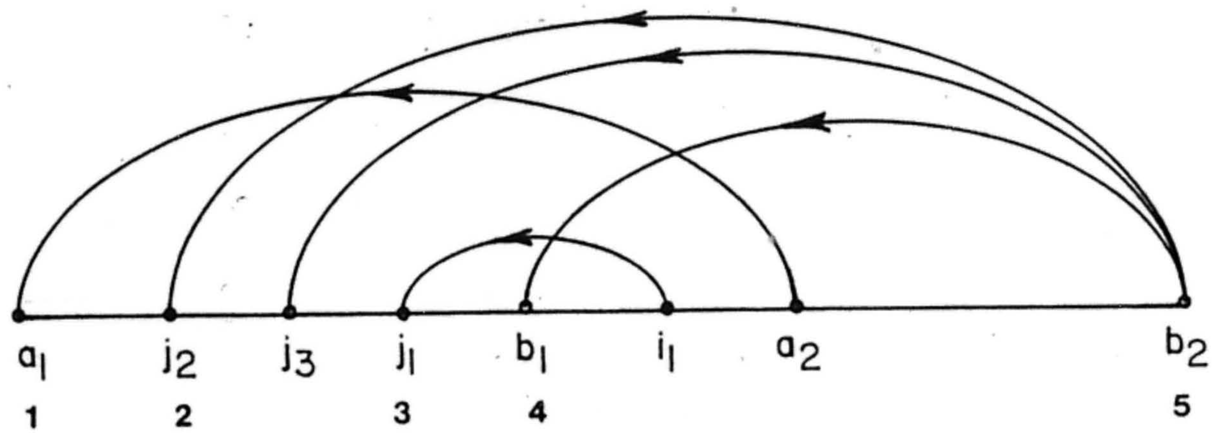


Figure 5.110

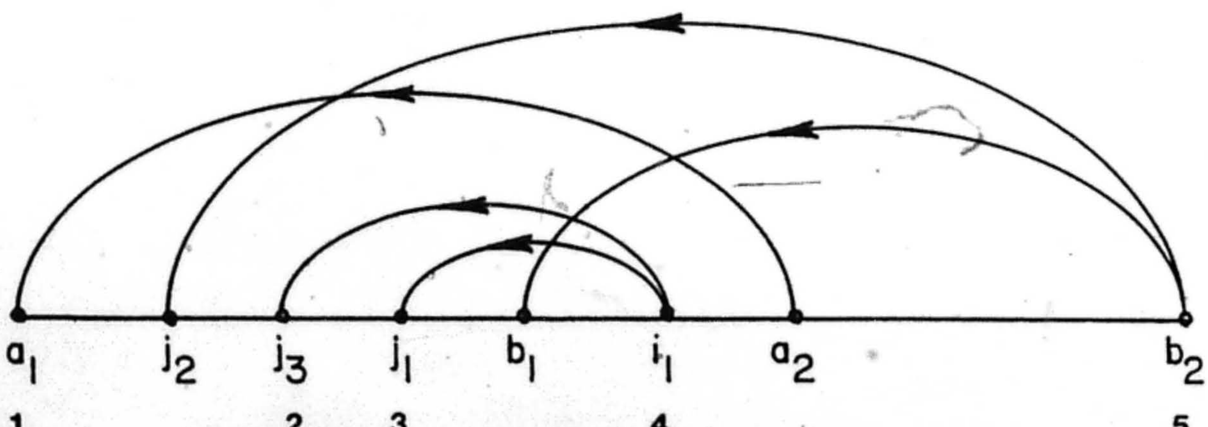


Figure 5.111

(i) Edge i_3j_3 is a C1(b) edge, but edge i_2j_2 is a C2(a) edge. In this case G mimics MIN.81. (See Figure 5.112)

(iii) Edge i_2j_2 is a C2(b) edge and edge i_3j_3 is a C1(b) edge. In this case a reduction of G is mimicked by MAX.3. (See Figure 5.113)

Case (5.5) is subdivided. Every edge here will be a type C1 edge. We may assume that these edges do not cross, for otherwise suppose that $j_2 < j_3 < i_2 < i_3$. Then G mimics MIN.33.
(See Figure 5.114)

(i) Edges i_2j_2, i_3j_3 are C1(a) edges. In this case G mimics MIN.54. (See Figure 5.115)

(ii) Edge i_2j_2 is a C1(a) edge and edge i_3j_3 is a C1(b) edge. We make a distinction:

$j_3 < j_2 = j_1 < i_1 < i_2 = i_3$. In this case G mimics MIN.37. (See Figure 5.116)

$j_3 = j_2 < j_1 < i_1 = i_2 < i_3$. In this

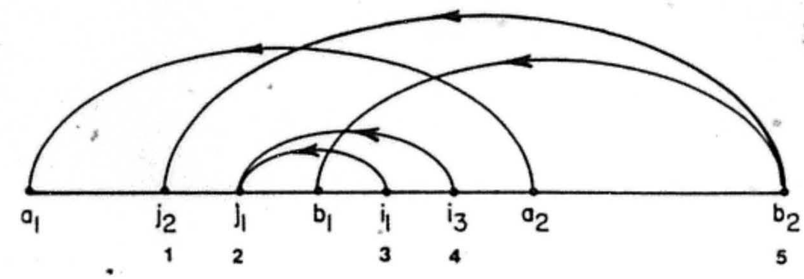


Figure 5.112

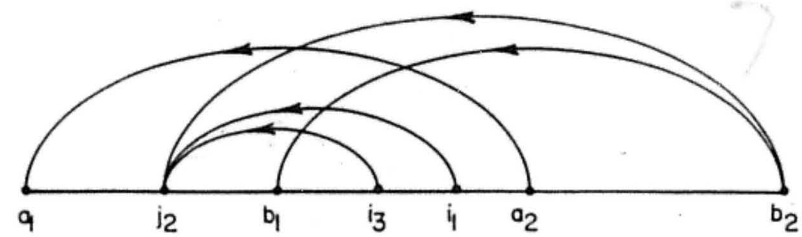


Figure 5.113

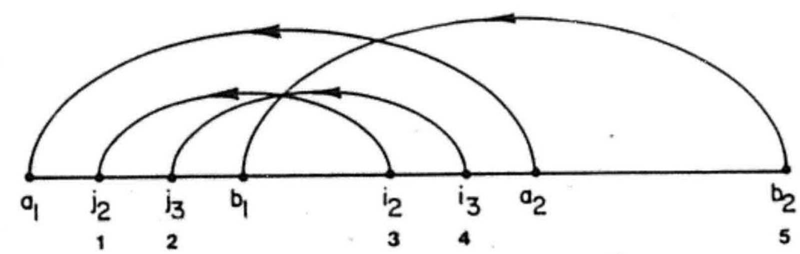


Figure 5.114

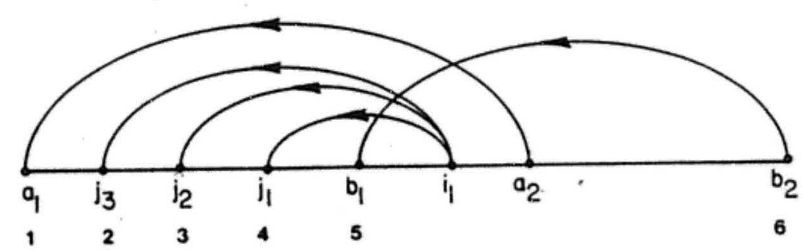


Figure 5.115

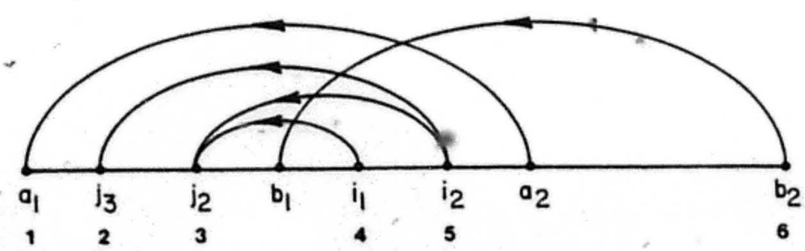


Figure 5.116

case G mimics MIN.37. (See Figure 5.117)

(iii) Edges i_3j_3 and i_2j_2 are C1(b) edges, viz $j_1 = j_2 = j_3$. In this case a reduction of G is mimicked by MAX.2.
(See Figure 5.118)

This completes the case where m = 3.

<u>$m > 3$:</u>

In view of the previous section, we may assume that $G \setminus i_4j_4$ falls under one of cases C5 + C3(b) (ii), C3(b) + C3(b), C3(b) + C2(b), C2(b) + C1(b) or C1(b)+ C1(b). We consider these cases one by one:

<u>$G \setminus i_4j_4$ falls under C5 + C3(b) (ii):</u>

We may also assume that $G \setminus i_3j_3$ falls under case C5 + C3(b), so that i_4j_4 is also a C3(b) edge. Say without loss that $i_1 < j_3 < j_4 < i_3 = i_4$. However, now j_3 is a vertex of P between i_1 and j_4, and $G \setminus i_3j_3$ falls under case C5 + C3(b) (i).

From now on assume that G has no edge of type C5.

<u>$G \setminus i_4j_4$ falls under C3(b) + C3(b):</u>

We may assume one of two alternatives:

$G \setminus i_3j_3$ falls under C3(b) + C3(b). In this case G mimics MIN.57. (See Figure 5.119)

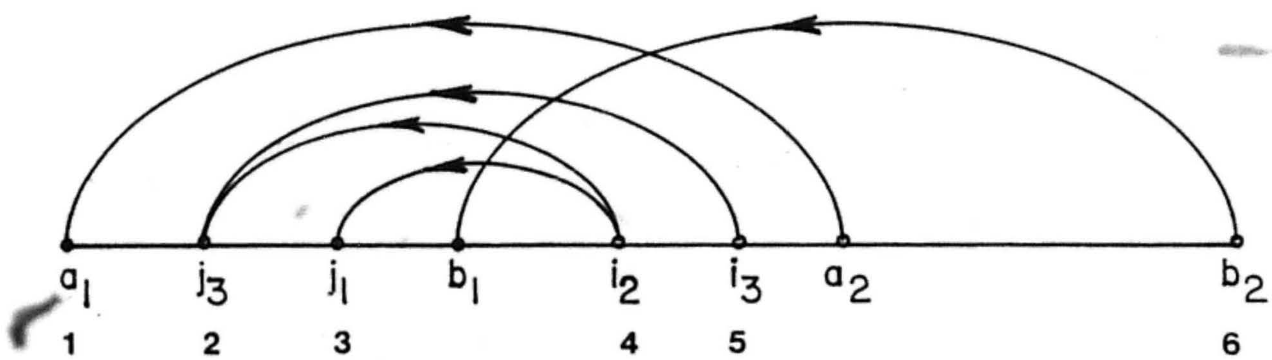


Figure 5.117

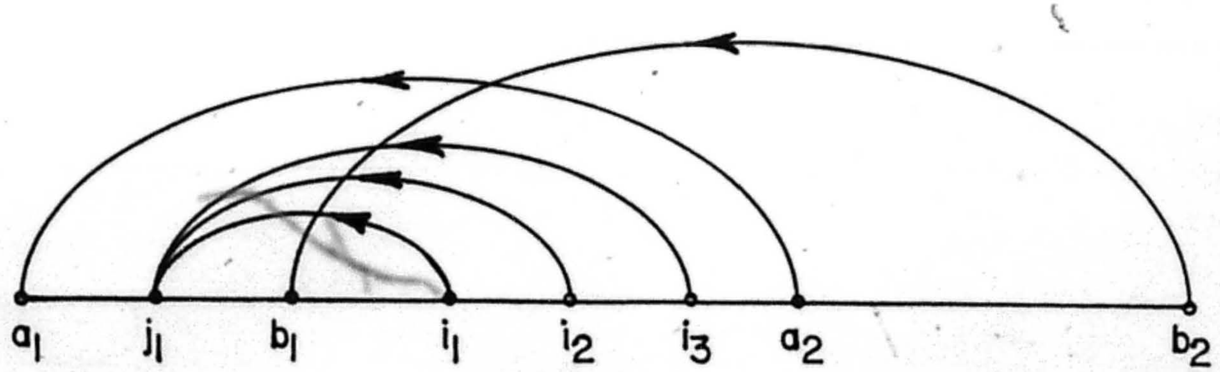


Figure 5.118

$G \setminus i_3j_3$ falls under C3(b) + C2(b). In this case G mimics MIN.58. (See Figure 5.120)

We assume from now on that G contains no type C3 edges.

<u>$G \setminus i_4j_4$ falls under C2(b) + C1(b):</u>

Here, if i_4j_4 falls into types C2 or C1(a), then $G \setminus i_3j_3$ falls under a case previously disposed of. We may therefore assume that i_4j_4 is an edge of type C1(b). Without loss of generality, say that $j_1 = j_3 = j_4 < i_4 < i_3 < i_1$. In this case G mimics MIN.59. (See Figure 5.121)

<u>$G \setminus i_4j_4$ falls under C1(b) + C1(b):</u>

Assume without loss of generality that i_4j_4 is a C1(b) edge, and $j_4 = j_3 = j_2 = j_1 < i_1 < i_2 < i_3 < i_4$. In this case G mimics MIN.60. (See Figure 5.122)

This concludes our classification, and the proof of this chapter's theorem.

Figure 5.119

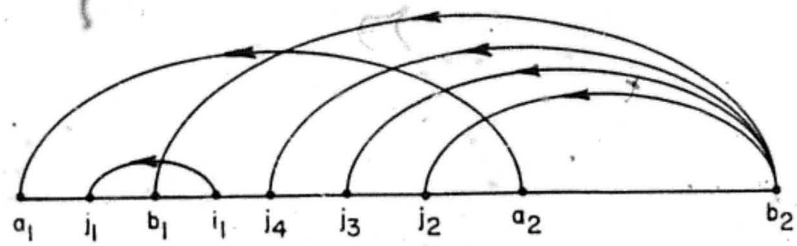


Figure 5.120

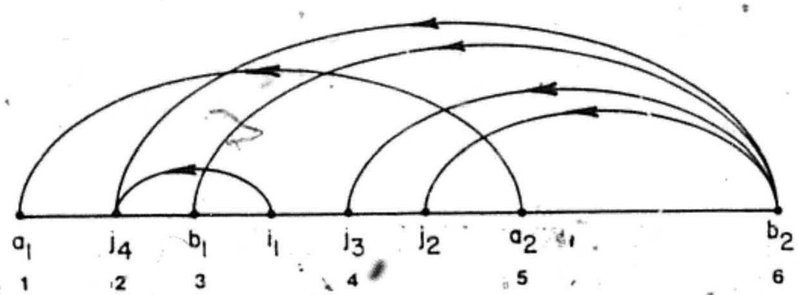


Figure 5.121

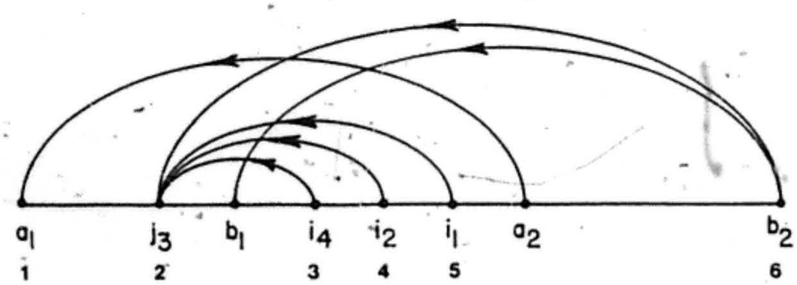
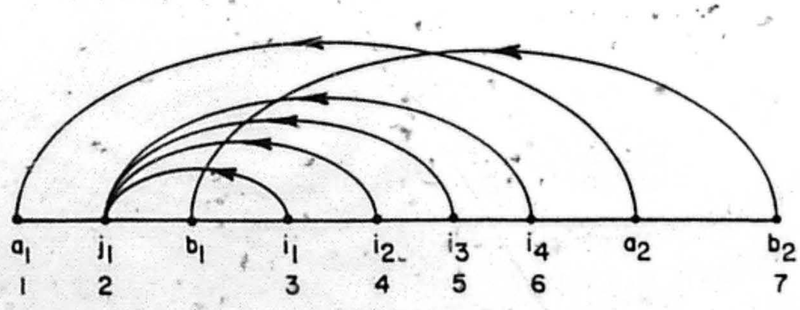


Figure 5.]22



Chapter 6: One Hump Digraphs

Definition: Let G be a one hump digraph. Then by the definition of one hump digraphs, we can write vert(G) = vert(P) where P is a directed Hamiltonian path in G. Also G has at least one additional edge c_1d_1 where c_1 is the terminal vertex of P, d_1 the initial vertex of P. Call $P \cup c_1d_1$ the skeleton of G. Any edge of G which is not in the skeleton is called an extra-skeletal edge of G.

Lemma 6.1: Let G be a one hump digraph with no useless edges, not mimicking a digraph in MIN. We may choose a skeleton $P \cup c_1d_1$ for G so that with respect to P, every extra-skeletal edge of G is a back edge.

Proof: Let Q be the subgraph of $G \setminus c_1d_1$ induced by P. Let J be the strongly connected component of Q containing c_1. Let us suppose that we have chosen P to make $| J |$ as large as possible. We will show that in this case, every extra-skeletal edge of G is a back edge with respect to P.

Suppose that G has an edge kl which is a forward edge with respect to P.

Case 1: We have $k, l \in$ vert(J).

Then there is a cycle C in J containing the edge kl. But the intersection of cycles $P \cup c_1d_1$ and C is not connected, so G mimics MIN.1 or MIN.2 by the Intersection

Lemma. (See Figure 6.1)

Case 2: We have $k \in$ vert($Q \setminus J$), $l \in$ vert(J). Then we find a new skeleton for Q replacing c_1d_1 by the edge km, where m is the successor of k in P, and replacing P by the path P' in $P \cup c_1d_1$ from m to k. Then the strongly connected component of $G \setminus km$ contains l, hence all of $J \cup \{ k \}$, contradicting our choice of P. (See Figure 6.2)

Case 3: We have k, $l \in$ vert($Q \setminus J$). By the last two cases, we may suppose that every forward edge of G with respect to P has both ends in $Q \setminus J$. However, by the maximality of J, we may assume that J contains more than one vertex, for if we pick P to end at k, then J contains { k, l }. Thus J has a back edge, and contains some M, since G has no useless edges. Then G mimics MIN.18. (See Figure 6.3.)

We therefore conclude that every edge of G is a back edge with respect to P.□

Proof of Theorem 3.10: The proof of this theorem, which takes up the body of this chapter, procedes by classifying the one hump digraphs. As usual, assume that

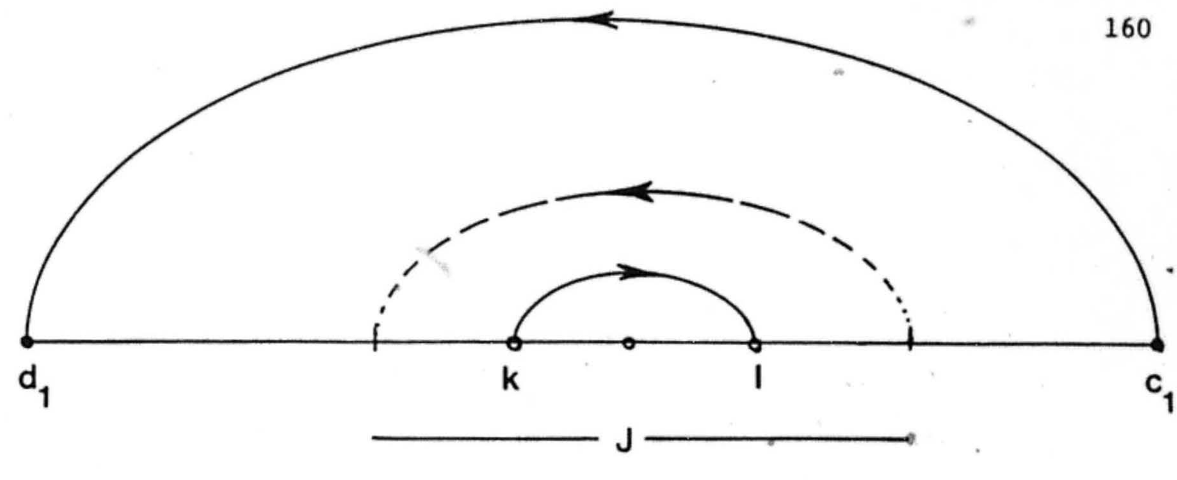


Figure 6.1

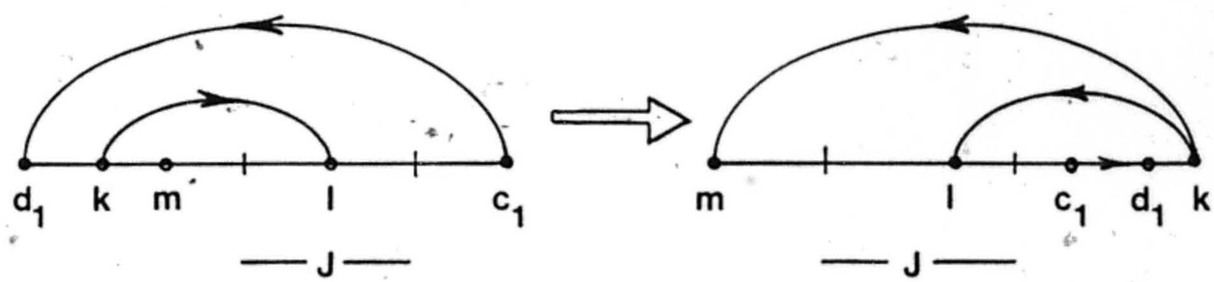


Figure 6.2

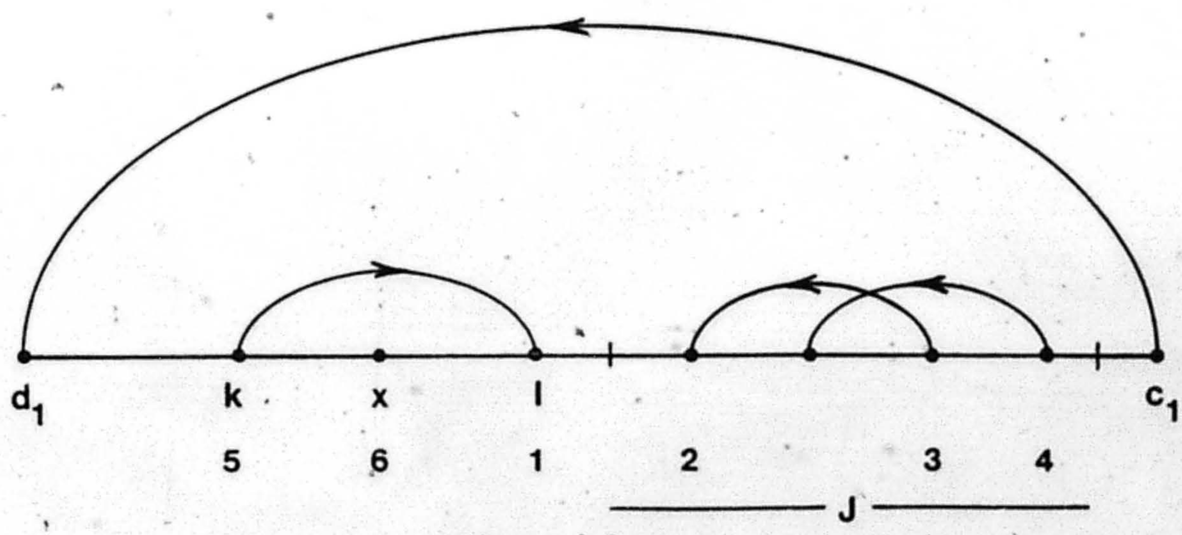


Figure 6.3

G is without useless edges. We can assume that every extra-skeletal edge of G is a back edge.

It is in this chapter that the "stripping" procedure introduced in the last chapter comes into its own. "Stripping" away edge c_1d_1 from G, we form cases based on the strongly connected components of $G \setminus c_1d_1$. A strongly connected component of $G \setminus c_1d_1$ which contains more than one vertex is called a <u>bubble</u>. We can assume that G has at least one bubble, since G contains an M. Our first level of subdivisions in the one hump case depends on the number of bubbles in G.

G has three or more bubbles

G has two bubbles

or G has only one bubble.

We look at these possibilities one by one:

<u>Subcase</u>: G contains three or more bubbles: Without loss of generality, we may assume that G has back edges i_1j_1, i_2j_2, i_3j_3, i_4j_4, i_5j_5, i_6j_6 where $j_1 < j_2 \le i_1 < i_2 < j_3 < j_4 \le i_3 < i_4 < j_5 < j_6 \le i_5 < i_6$. This is because each bubble of G must contain an M, as in Case 3 of Lemma 6.1. In this case G mimics MIN.61. (See Figure 6.4.)

<u>Subcase</u>: G contains two bubbles. Refer to the bubbles of G as C_1 and C_2 respectively. Without loss of

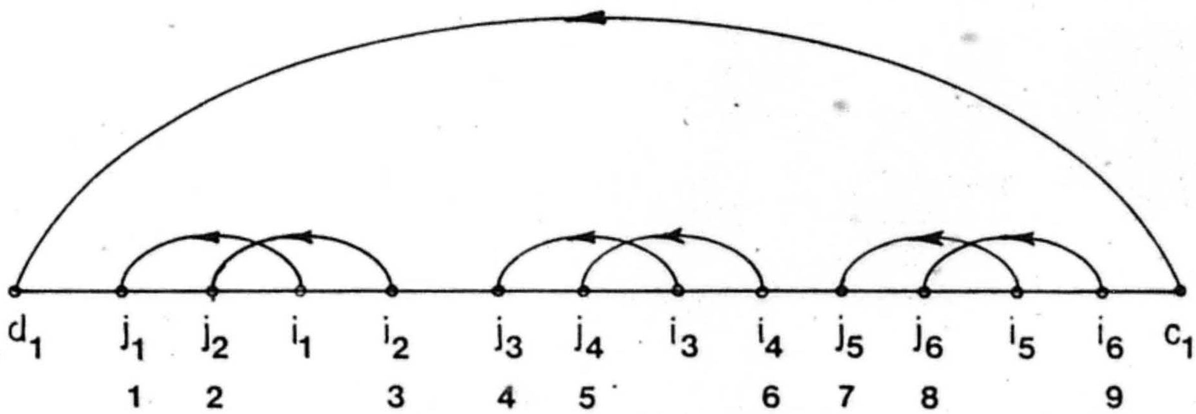


Figure 6.4

generality we may assume that each of C_1, C_2 is a one hump, two hump or three hump digraph, by the classification lemma. For the sake of definiteness, say that the vertices of C_1 precede those of C_2 on P. However, note that if we so desire, we can reverse the order of C_1 and C_2 on P by putting G into normal form in a different way: Simply rotate the skeleton of G. (See Figure 6.5.) Thus C_1 and C_2 are interchangeable. We now form cases based on C_1 and C_2.

<u>Subcase α</u>: One of C_1, C_2 is a three hump digraph. Assume without loss of generality that C_1 is a three hump digraph. In any case, C_2 contains an M. Therefore G has back edges i_1j_1, i_2j_2, i_3j_3, i_4j_4, i_5j_5 where $j_1 < j_2 \leq i_1 < j_3 \leq i_2 < i_3 < j_4 < j_5 \leq i_4 < i_5$. In this case G mimics MIN.62. (See Figure 6.6.)

<u>Subcase β</u>: Both of C_1, C_2 are two hump digraphs.

<u>Subcase γ</u>: One of C_1, C_2 is a two hump digraph, and the other is a one hump digraph.

<u>Subcase δ</u>: Both of C_1, C_2 are one hump digraphs.

The point of our "stripping" classification is to

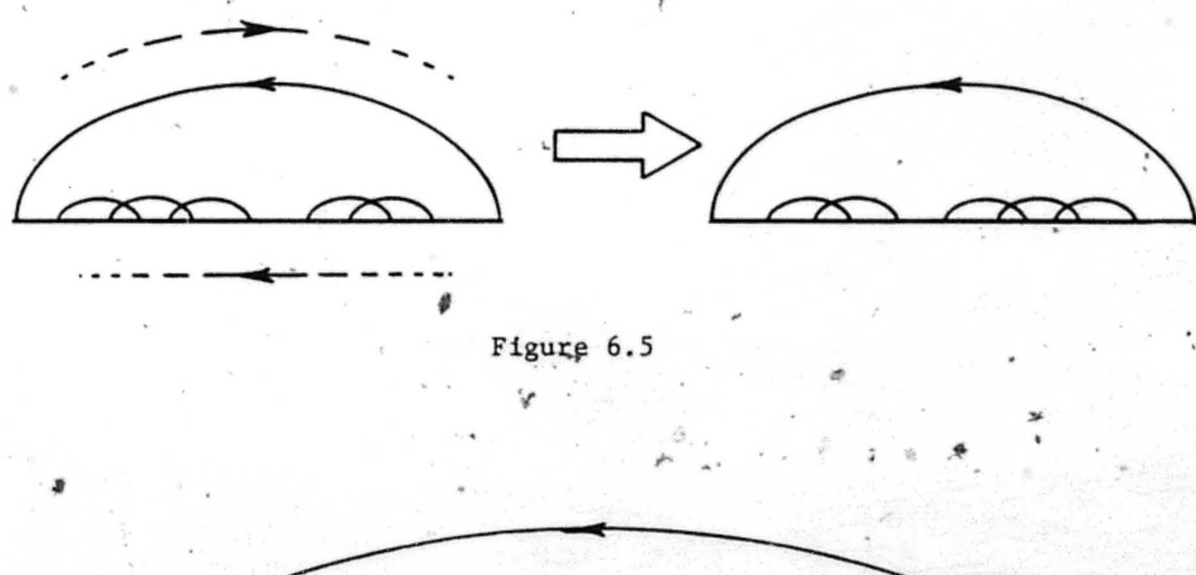


Figure 6.5

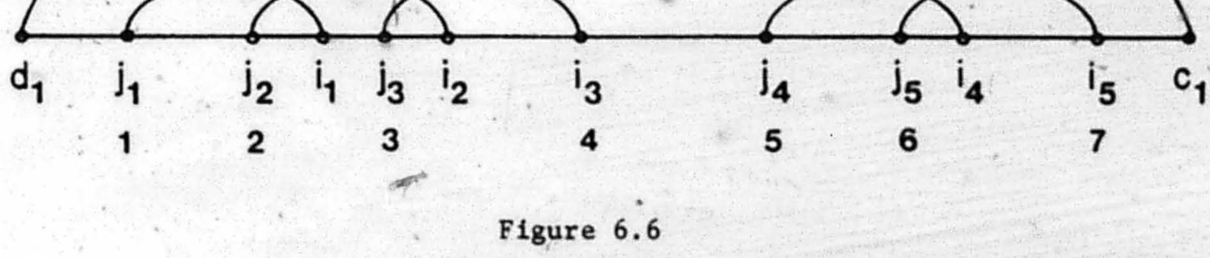


Figure 6.6

make use here of work done in the previous chapters. Consider the situation in subcase β. Here G has at least four extra-skeletal edges a_2a_1, b_2b_1, $a_2'a_1'$, $b_2'b_1'$ where the skeleton of C_1 consists of the edges a_2a_1, b_2b_1 and the path in P from a_1 to b_2, and the skeleton of C_2 consists of the edges $a_2'a_1'$, $b_2'b_1'$ and the path in P from a_1' to b_2'. Any further extra-skeletal edges of G appear as extra-skeletal edges in C_1 or C_2, and this leads to the following subdivision of subcase β:

<u>Subcase β.1</u>: G has exactly four extra-skeletal edges

Recall that the skeleton of G is a cycle. This cycle gives a circular order to the vertices of G. We have two possibilities:

(i) No vertex of C_1 is a predecessor of a vertex of C_2 in the circular order, and no vertex of C_2 is a predecessor of a vertex of C_1 in the circular order. In this case, without loss of generality up to rotation of the skeleton, we may assume that G has a vertex x where $b_2 < x < a_1'$, and that $d_1 < a_1$. Then G mimics MIN.63. (See Figure 6.7.)

(ii) Case (i) does not occur. Thus if G has a vertex x where $b_2 < x < a_1'$ then $d_1 = a_1$ and $b_2' = c_1$. In this case, a reduction of G is mimicked by MAX.26. (See

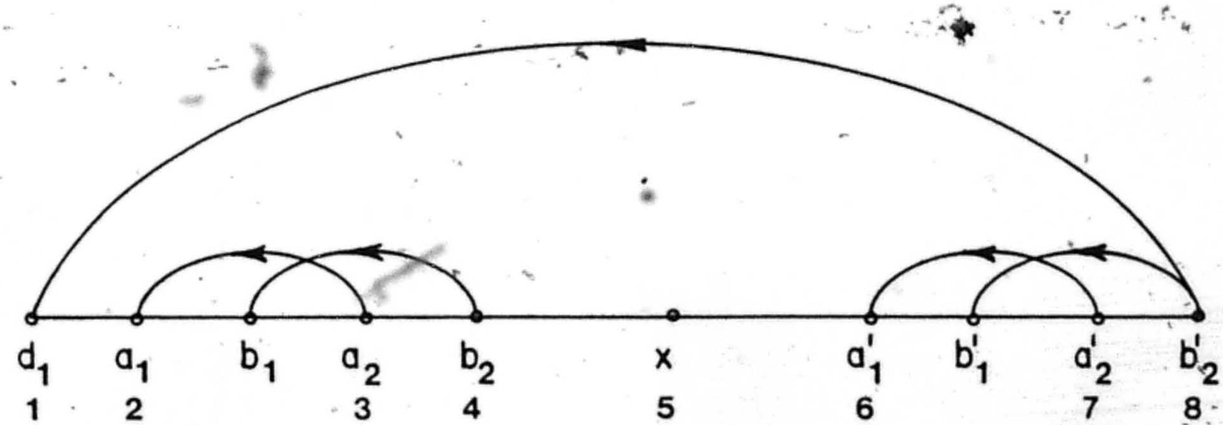


Figure 6.7

Figure 6.8.)

Subcase β.2: G has a fifth extra-skeletal edge.

Without loss of generality, (up to rotation and reversal), C_1 has one or more extra-skeletal edges falling into the categories of the previous chapter. This gives five possibilities:

(i) C_1 has an edge of type B of the previous chapter. In this case G mimics MIN.64. (See Figure 6.9.)

(ii) C_1 has an edge of type C of the previous chapter. In this case G mimics MIN.65. (See Figure 6.10.)

(iii) C_1 has an edge of type D of the previous chapter. In this case G mimics MIN.65. (See Figure 6.11.)

(iv) C_1 has no edges of types B, C, D, however, C_1 does have an edge of type E of the previous chapter. We may thus assume that C_1 has two type E edges forming an M, and, as in the previous chapter, C_1 mimics MIN.15. (See Figure 6.12)

Figure 6.8

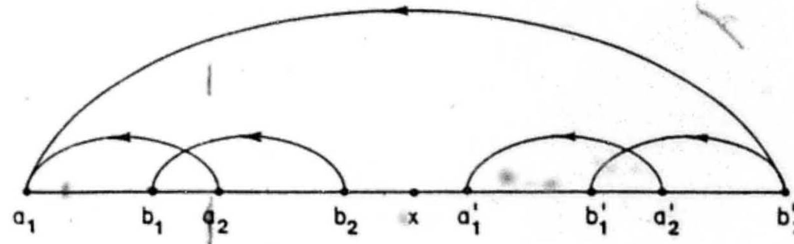


Figure 6.9

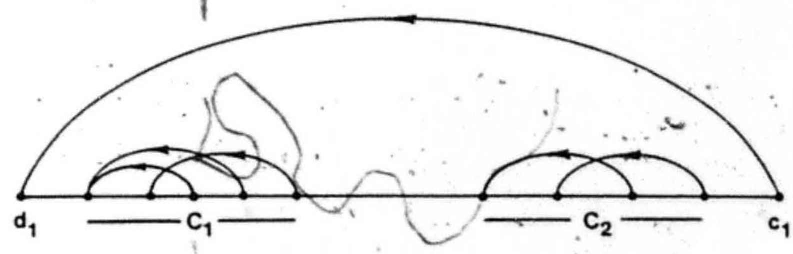


Figure 6.10

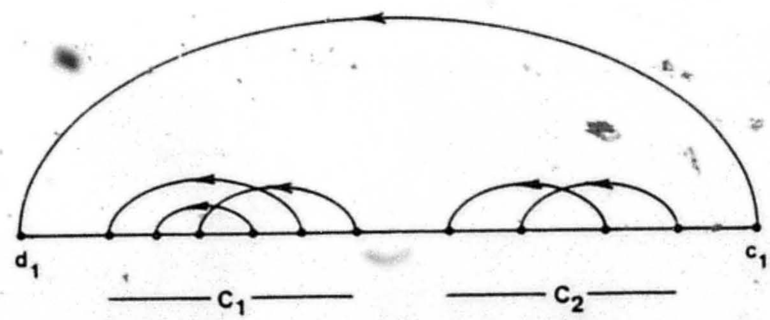
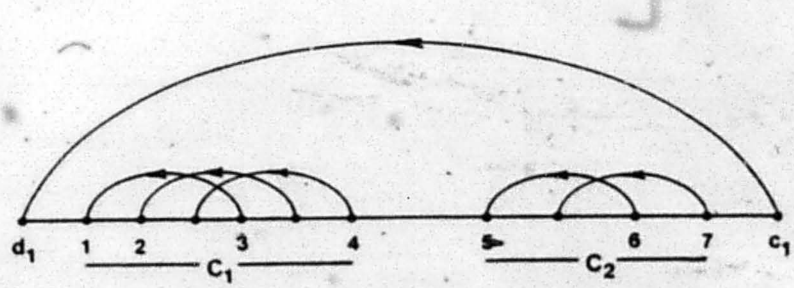


Figure 6.11



(v) C_1 has no edges of types B, C, D, however, C_1 does have an edge of type A of the previous chapter. We may thus assume that C_1 has two type A edges forming an M. In this case, G mimics MIN.66. (See Figure 6.13.)

This concludes our consideration of subcase β. We next consider subcase γ. Without loss of generality, assume that C_1 is a one hump digraph, and C_2 is a two hump digraph. Repeatedly apply our stripping procedure to C_1. Eventually we arrive at a digraph C_1' which is a two hump or three hump digraph. If C_1' is a three hump digraph, then G mimics a digraph of MIN as in subcase α. Therefore assume without loss of generality that C_1' is a two hump digraph.

By our examination of subcase β, we may assume that neither C_1' nor C_2 has extra-skeletal edges. Then without loss of generality, using rotations and reflections, the structure of G is as follows:

G has extra-skeletal edges

$$a_2a_1,\ b_2b_1,\ a_2'a_1',\ b_2'b_1',\ i_1j_1,\ i_2j_2,\ \dots,\ i_sj_s$$

with $a_1 < b_1 \le a_2 < b_2 < a_1' < b_1' \le a_2' < b_2'$

$$j_s \le j_s \le \dots \le j_1 \le a_1 < b_2 \le i_1 \le \dots$$

$$\le i_{s-1} \le i_s < a_1'.$$

and not both $j_1 = a_1$ and $i_1 = b_2$ (since otherwise edge

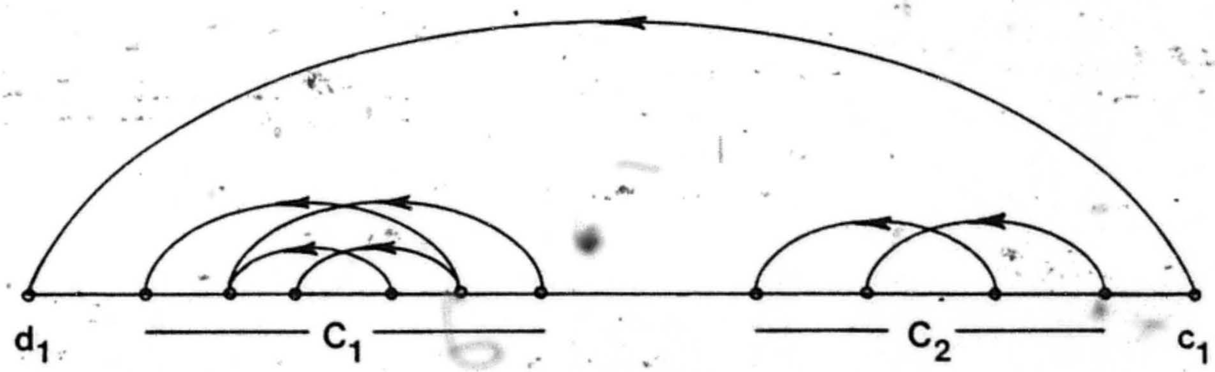


Figure 6.12

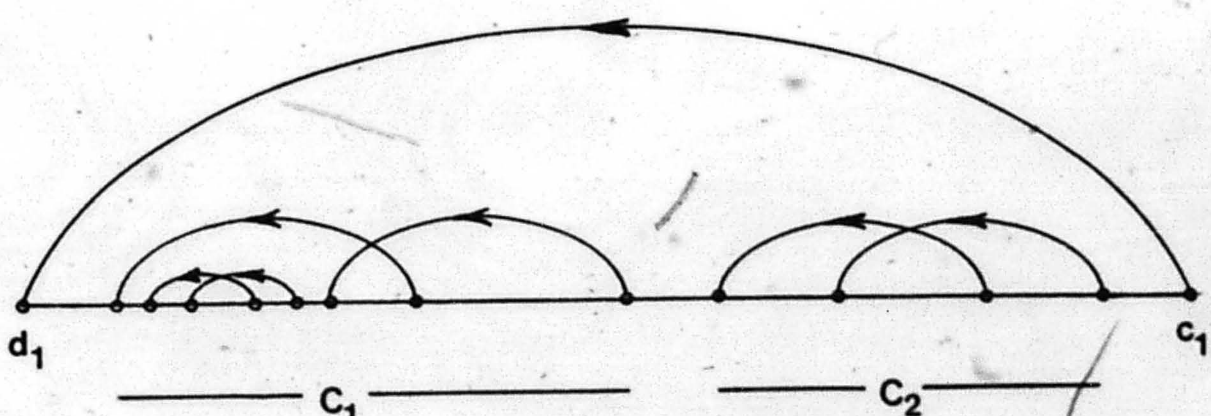


Figure 6.13

i_1j_1 would be useless, not properly containing the M formed by a_2a_1, b_2b_1). Without loss of generality, (again up to reversal and rotation of G), say that $j_1 \neq a_1$.

We make the following subcases.

<u>Subcase γ.1</u>: We have s = 1.

There are two possibilities.

(i) $i_1 \neq b_2$. In this case G mimics MIN.18. (See Figure 6.14.)

(ii) $i_1 = b_2$. If G has a vertex x between b_2 and a_1', then G mimics MIN.63. (See Figure 6.15.)
If G has no vertex between b_2 and a_1', then a reduction of G is mimicked by MAX.26. (See Figure 6.16.)

<u>Subcase γ.2</u>: We have s > 1.

By subcase γ.1, we may assume that $i_2 = i_1 = b_2$, and thus that $j_2 < j_1$. In this case G mimics MIN.67. (See Figure 6.17.)

This concludes our examination of subcase γ.

We next consider subcase δ. We may assume without

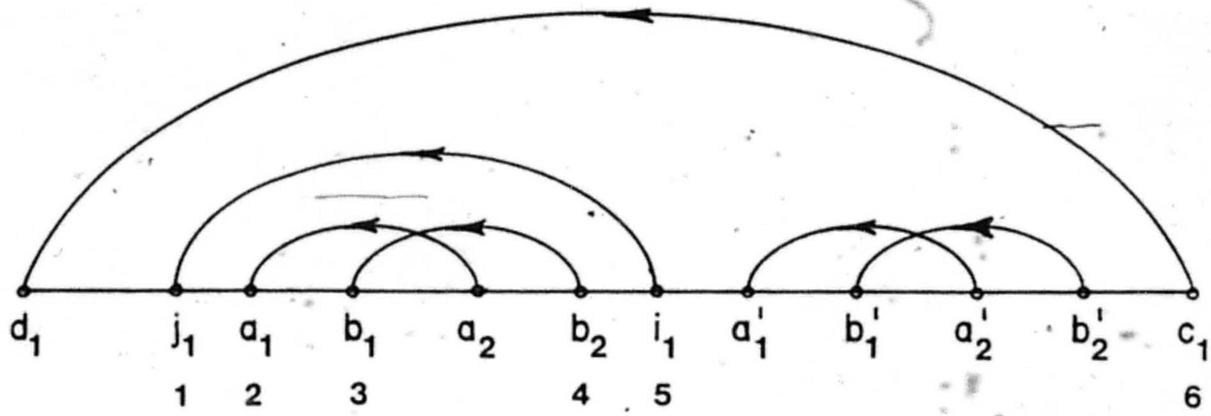


Figure 6.14

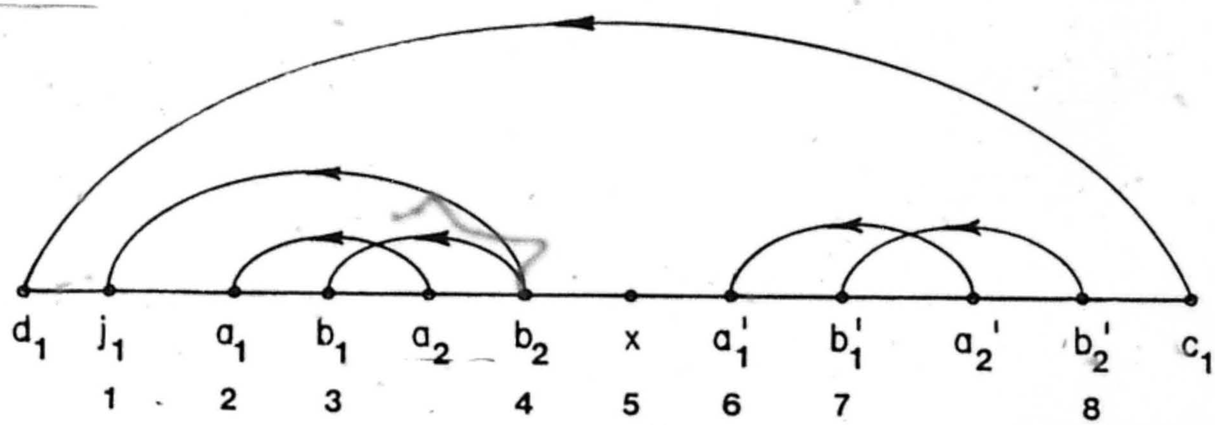


Figure 6.15

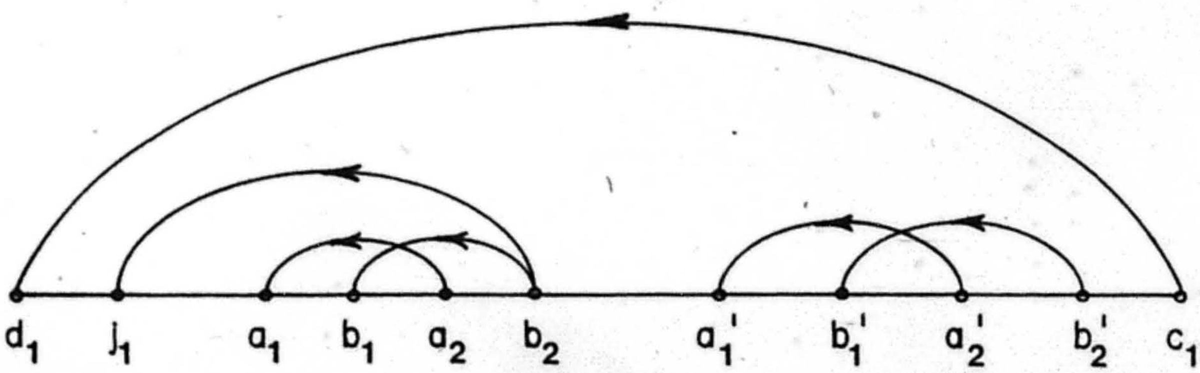


Figure 6.16

loss of generality that after iterated stripping C_1 and C_2 are two hump digraphs. Using rotation and reflection and subcase γ, we may say without loss of generality that one of the following two cases occurs:

<u>Subcase δ.1</u>: G has extra-skeletal edges a_2a_1, b_2b_1, $a_2'a_1'$, $b_2'b_1'$, i_1j_1', i_2j_2 where

$j_1 < a_1 < b_1 \le a_2 < b_2 = i_1 < j_2 < a_1' < b_1' \le a_2' < b_2' = i_2$

In this case G mimics MIN.63. (See Figure 6.18.)

<u>Subcase δ.2</u>: G has extra-skeletal edges a_2a_1, b_2b_1, $a_2'a_1'$, $b_2'b_1'$, i_1j_1, i_2j_2 where

$j_1 = a_1 < b_1 \le a_2 < b_2 < i_1 < j_2 < a_1' < b_1' \le a_2' < b_2' = i_2$

In this case G mimics MIN.68. (See Figure 6.19.)

This concludes our examination of subcase δ, and hence our consideration of the case when G has exactly two bubbles.

<u>Subcase</u>: G has exactly one bubble:

Let the bubble be called C_1. If C_1 is a one hump digraph, then suppose that under C_1 we have two disjoint M's: viz. G has edges a_2a_1, b_2b_1, $a_2'a_1'$, $b_2'b_1'$, i_1j_1, i_2j_2 where

$j_2 \le j_1 \le a_1 < b_1 \le a_2 < b_2 < a_1' < b_1' \le a_2' < b_2' \le i_1 \le i_2$

and $d_1 = j_2$, $c_1 = i_2$. By rotation, assume that $d_1 = j_1$.

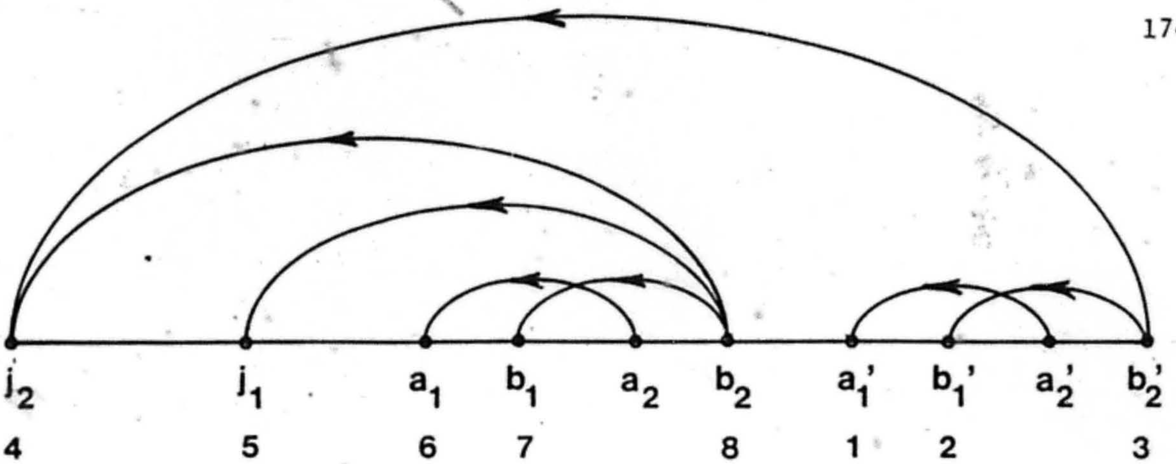


Figure 6.17

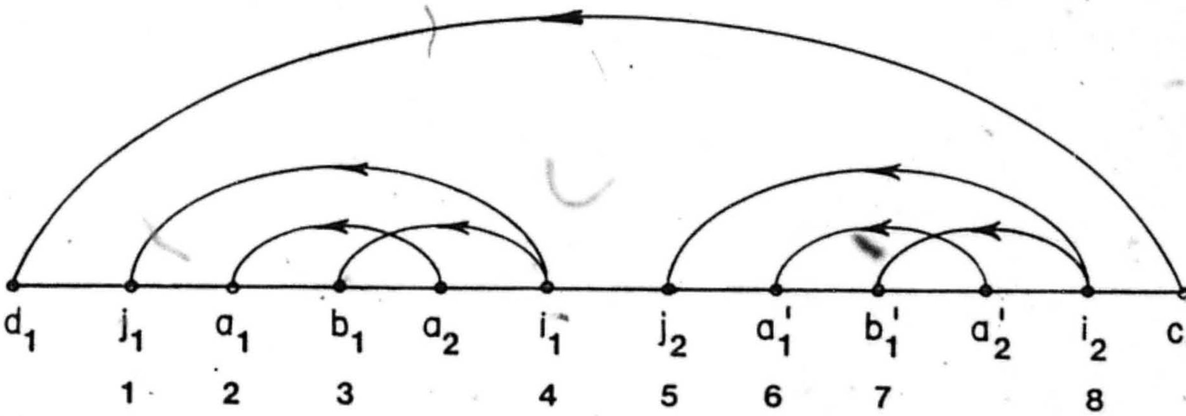


Figure 6.18

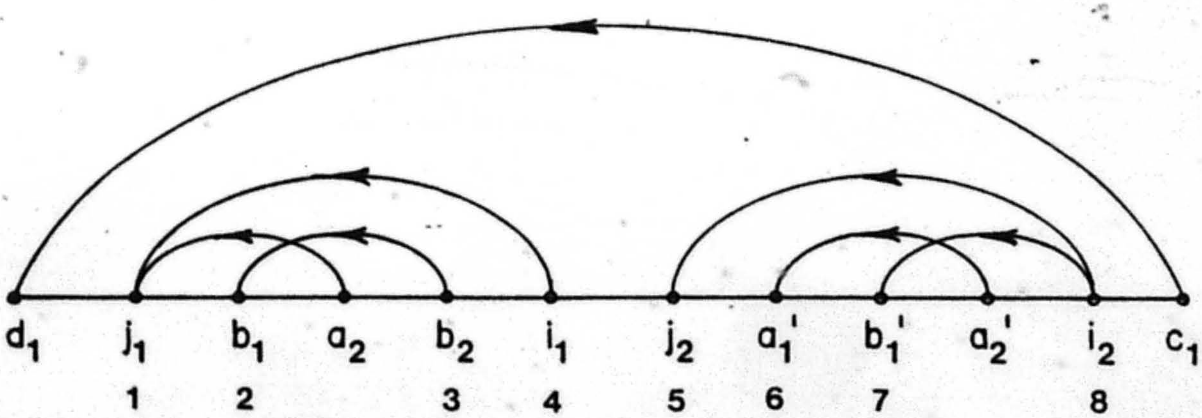


Figure 6.19

In this case, G mimics MIN.69. (See Figure 6.20.)

We may thus suppose without loss of generality that whenever G is a digraph with only one bubble, by repeated iteration of the stripping procedure on C_1 we will eventually arrive at a digraph C_1' which is a two or three hump digraph. This justifies the following case breakdown:

<u>Subcase α</u>: C_1' is a three hump digraph.
<u>Subcase β</u>: C_1' is a two hump digraph.

In pursuing subcase α, by our classification of the three hump case, we may assume without loss of generality that either (i) C_1' falls under case A2(b), $j = b_1$ of the classification of the three hump digraphs
or (ii) C_1' is its own skeleton.

In subcase α (i), G mimics MIN.70. (See Figure 6.21.)

In subcase α (ii), we make a distinction, depending on whether $\text{vert}(G) = \text{vert}(C_1')$.

If $\text{vert}(G) = \text{vert}(C_1')$ then a reduction of G is mimicked by MAX.25. (See Figure 6.22.)

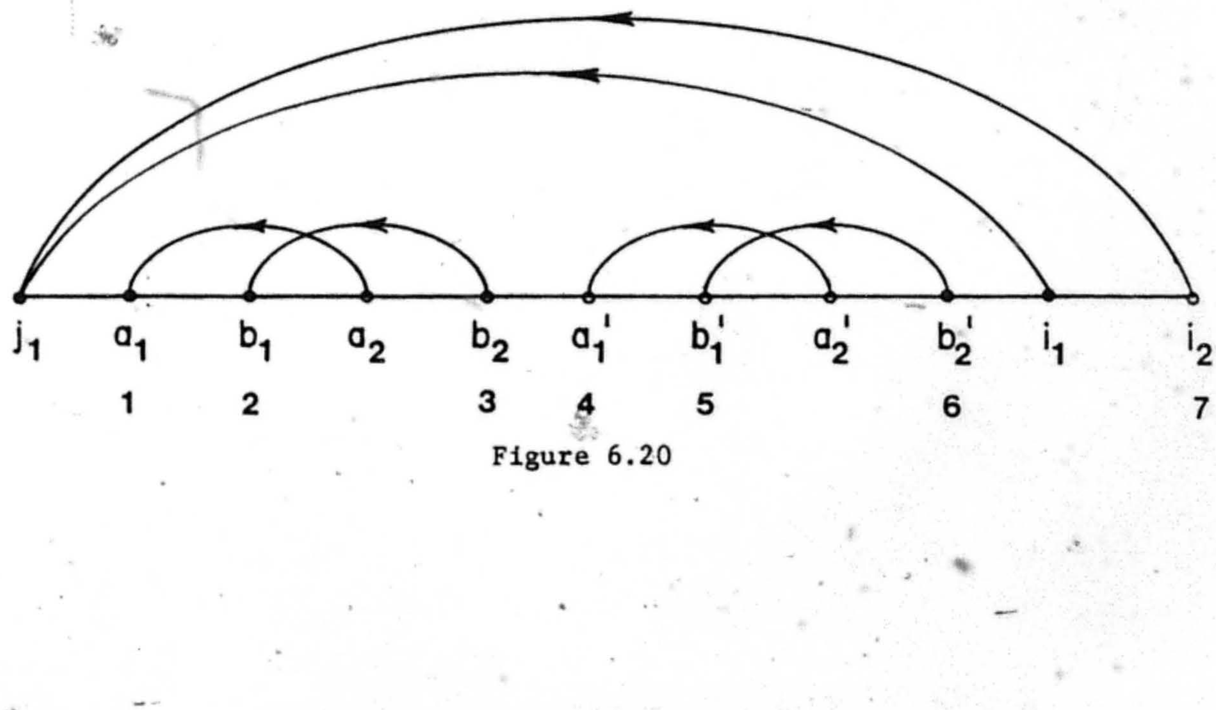


Figure 6.20

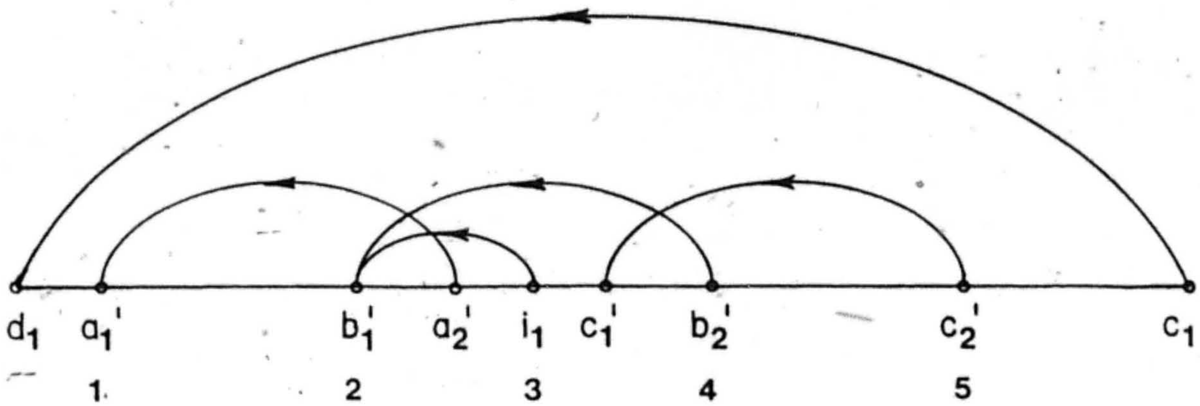


Figure 6.21

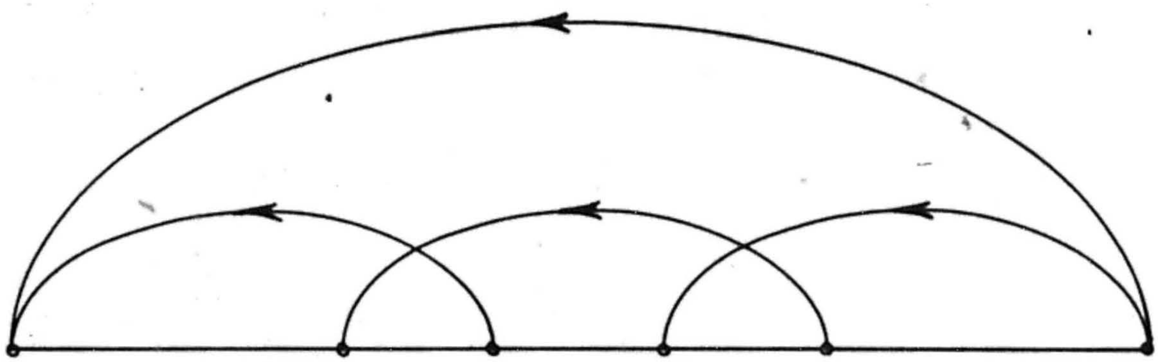


Figure 6.22

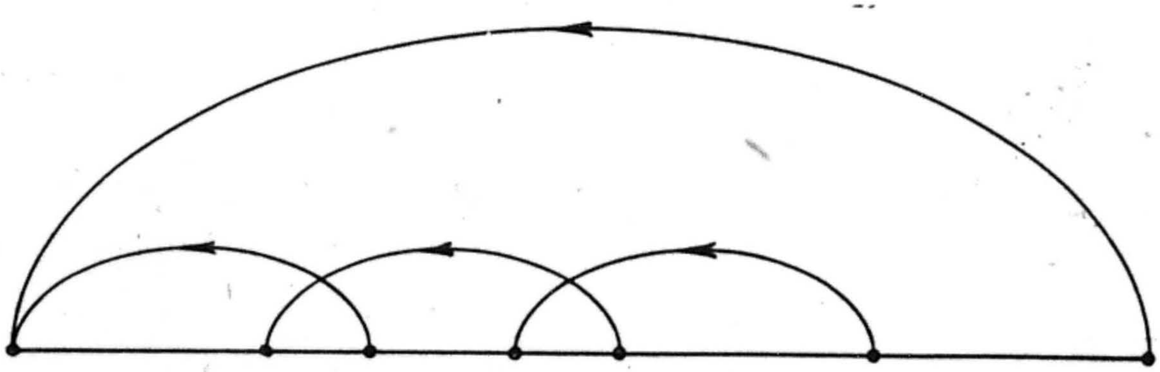


Figure 6.23

Otherwise, G mimics MIN.71. (See Figure 6.23.)

We now turn to subcase β.

In this case, G has (at least) certain edges a_2a_1, b_2b_1, i_1j_1, i_2j_2, ..., i_sj_s where $d_1 = j_s \leq j_{s-1} \leq \dots \leq j_1 \leq a_1 < b_1 \leq a_2 < b_2 \leq i_1 \leq \dots \leq i_s = c_1$. Here a_2a_1, b_2b_1, along with the piece of P from a_1 to b_2 form the skeleton of C_1'. The edges i_rj_r are those that were stripped from G to arrive at C_1'. We now make cases depending on the form of C_1'.

Subcase- C_1' is its own skeleton: In this case, since edge i_1j_1 is not useless, we cannot have both $j_1 = a_1$ and $i_1 = b_2$. Without loss of generality (up to reflection), say that $i_1 \neq b_2$. We make further subdivisions based on the ij edges.

Subcase- s = 1: In this case, a reduction of G is mimicked by MAX.26. (See Figure 6.24.)

Subcase- s = 2: We have two possibilities. (i) $j_1 \neq a_1$. Without loss of generality, $i_1 \neq i_2$. In this case G mimics MIN.18. (See Figure 6.25.)

(ii) $j_1 = a_1$. In this case (without loss of generality up to a rotation), $j_2 = a_1$ also. Then a reduction of G is mimicked by MAX.24. (See Figure 6.26.)

<u>Subcase- s = 3</u>: Because of our observations in the s = 2 case, we may now without loss of generality assume that $j_3 = j_2 = j_1 = a_1$. Then a reduction of G is mimicked by MAX.24. (See Figure 6.27.)

<u>Subcase- s > 3</u>: Here G mimics MIN.72. (See Figure 6.28.)

This finishes the case where C_1' is its own skeleton. Returning to the theme of our stripping procedure, we use the two hump classification of the previous chapter on C_1'.

<u>Subcase- C_1' has an edge of type D</u>: Here G mimics MIN.35. (See Figure 6.29.)

From now on assume that G has no edges of type D.

<u>Subcase- C_1' has edges of type E only</u>: Here C_1' must have two type E edges forming an M, and C_1' mimics MIN.15 as we have already seen.

Figure 6.24

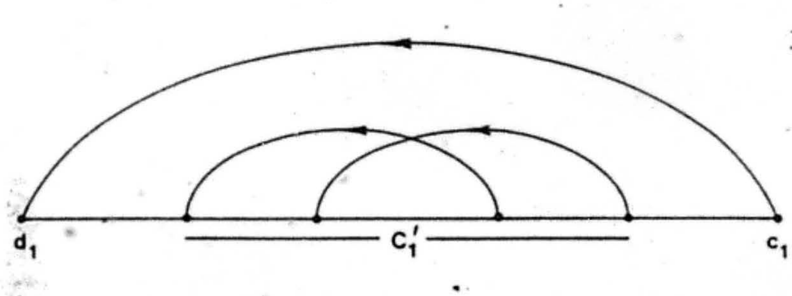


Figure 6.25

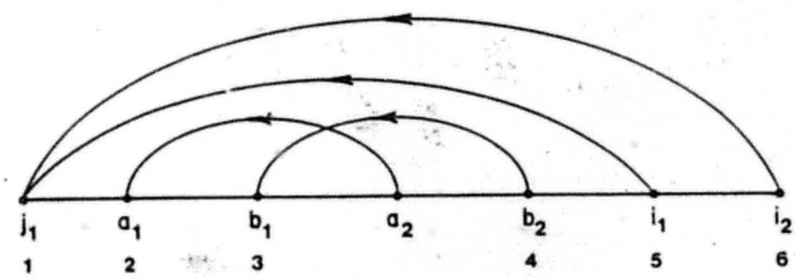


Figure 6.26

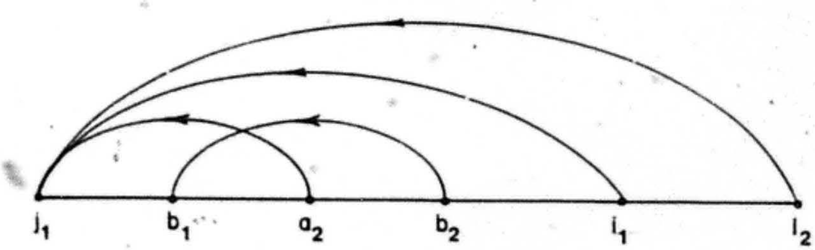


Figure 6.27

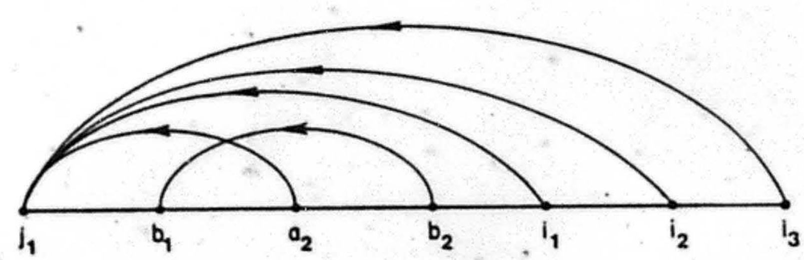
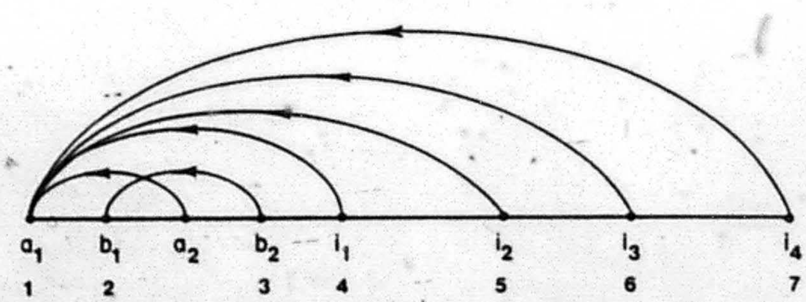


Figure 6.28



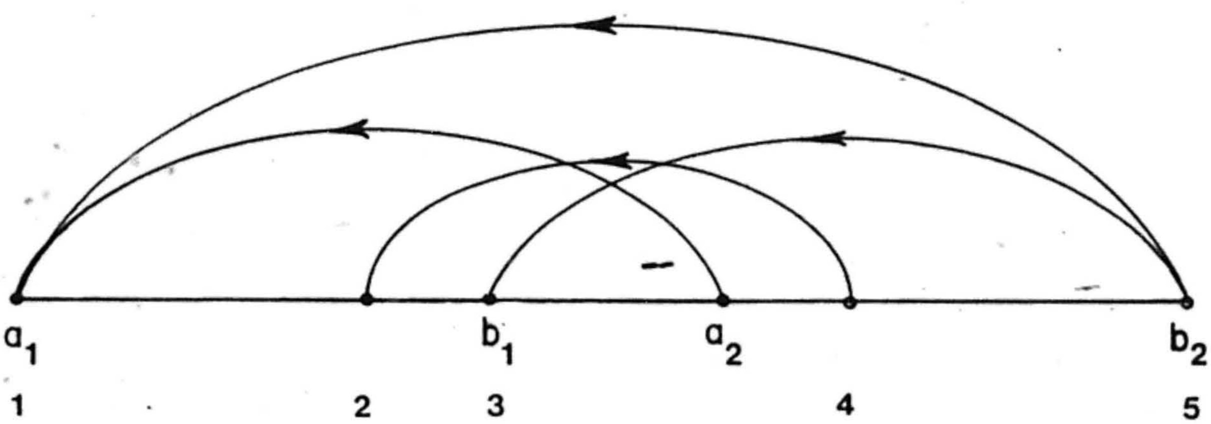


Figure 6.29

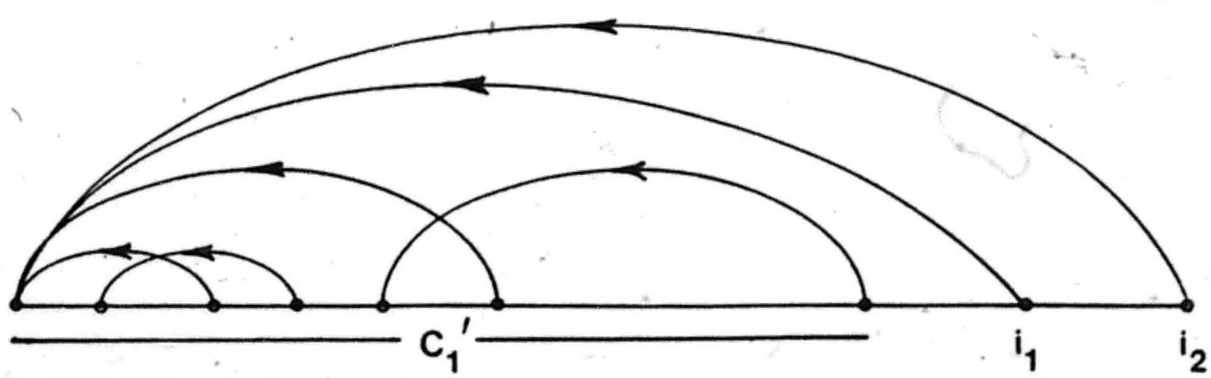


Figure 6.30

<u>Subcase- C_1' has edges of types A and E only</u>: We may assume by the foregoing subcase that C_1' has no edge of type E. Thus C_1' has two type A edges $k_1 l_1$, $k_2 l_2$ forming an M. By the case where C_1' is its own skeleton and $j_1 \neq a_1$, assume that

$a_1 = l_1 < l_2 \leq k_1 < k_2 < b_1$. (See Figure 5.22.) We make further subdivisions based on the ij edges.

<u>Subcase- $s > 1$</u>: In this case, G mimics MIN.66. (See Figure 6.30.)

<u>Subcase- $s = 1$</u>: If $i_1 \neq b_2$, then G mimics MIN.66. (See Figure 6.31.)

We assume from here on that $i_1 = b_2$. If C_1' has only two extra-skeletal edges, then a reduction of G is mimicked by MAX.23. (See Figure 6.32.)

If C_1' has a third extra-skeletal edge $k_3 l_3$, we may assume without loss of generality, by the classification done in the previous chapter, that $l_3 = a_1$, $k_3 > k_2$. (See Figure 5.33.) In this case, G mimics MIN.73. (See Figure 6.33.)

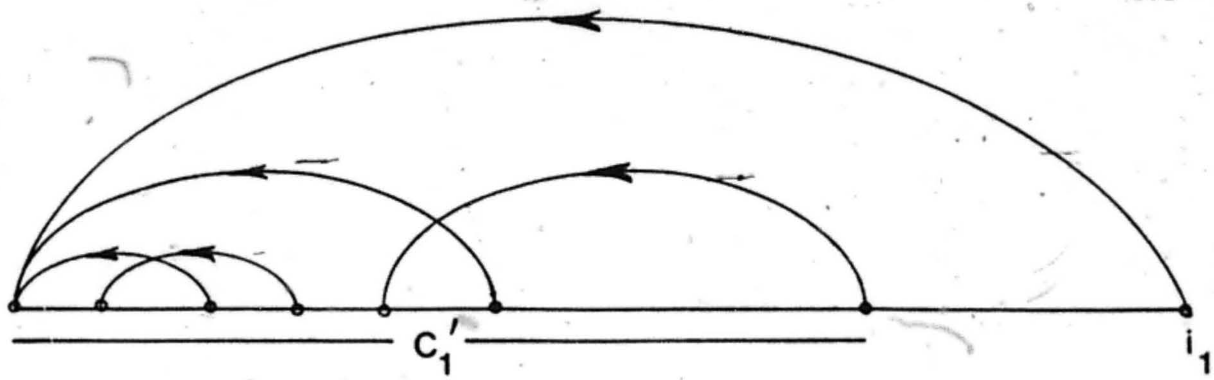


Figure 6.31

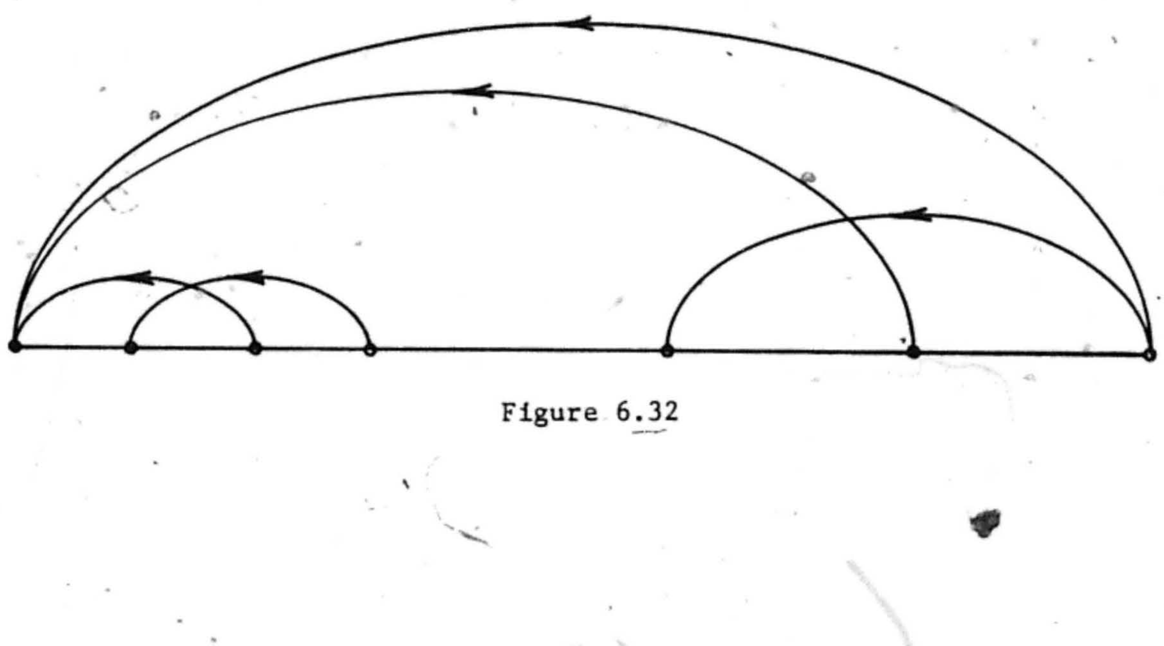


Figure 6.32

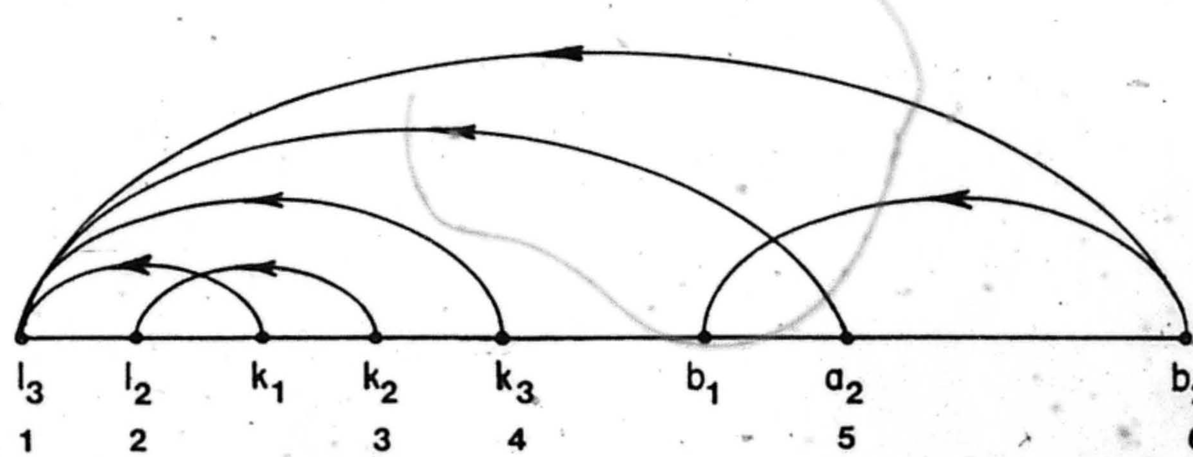


Figure 6.33

This completes those cases where all edges of C_1' are of types A, E or D of the previous chapter.

<u>Subcase- C_1' has an edge of type B, but no type C edges</u>: From our experience in the previous chapter, we may assume that <u>every</u> extra-skeletal edge of C_1' is either a normal type B edge, or a reversed type B edge where the normal and reversed edges never cross to form any M. Let the extra-skeletal edges of C_1' be

$k_1a_1, k_2a_1, \ldots, k_pa_1, b_2l_1, b_2l_2, \ldots, b_2l_q$ where $b_1 \leq k_1 < k_2 <\ldots k_p < l_q < l_{q-1} <\ldots < l_1 \leq a_2$, $p \geq q$.

We base our case division here on p, q and s:

<u>$q = 0$</u>:

<u>$s = 1$</u>:

Assume (by rotation) that $a_1 = j_1$.
If $i_1 = b_2$, then the edge i_1j_1 is useless, a contradiction. (See Figure 6.34.)

In fact, for the case where every edge of C_1' is a type B edge, we may assume that $a_1 = j_1$, $i_1 \neq b_2$.
If $i_1 \neq b_2$, then if $p \geq 3$, G mimics MIN.74. (See Figure 6.35.)

If $i_1 \neq b_2$ and if $p < 3$, then a reduction of G ~~is~~ mimicked by MAX.22. (See Figure 6.36.)

Figure 6.34

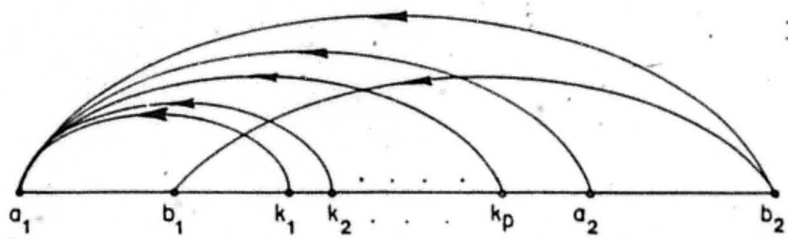


Figure 6.35

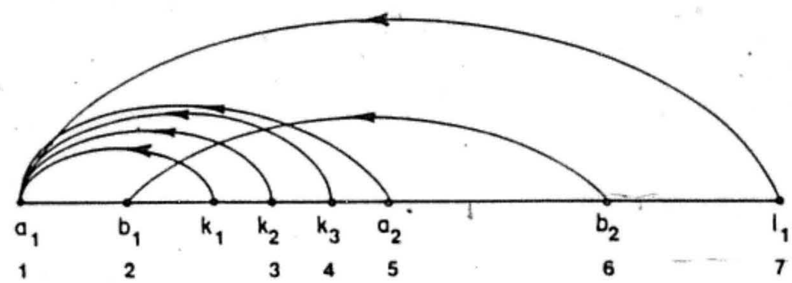


Figure 6.36

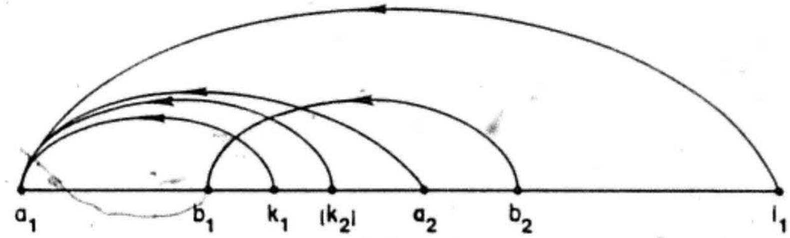


Figure 6.37

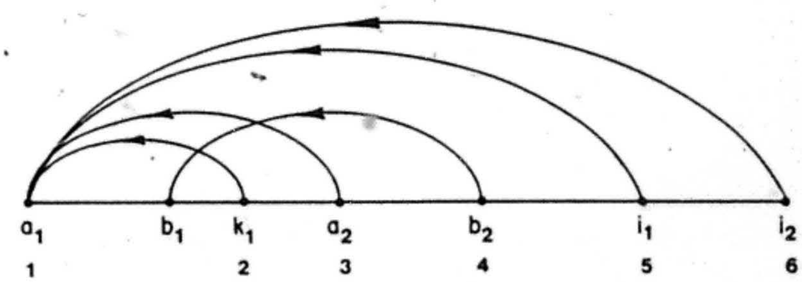
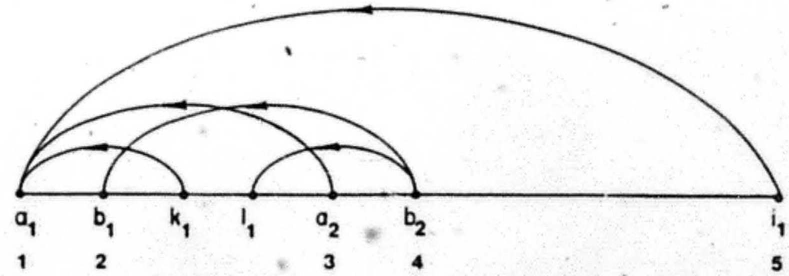


Figure 6.38



<u>$s > 1$</u>:

Again assume that $j_1 = j_2 = a_1$.

As $i_1 \neq b_2$, G mimics MIN.75. (See Figure 6.37.)

<u>$q > 0$</u>:

Assume that $j_1 = a_1$.
As $i_1 \neq b_2$, G mimics MIN.76. (See Figure 6.38.)

This finishes the case when C_1' has no type C edges.

<u>Subcase- C_1' has an edge of type C</u>: Call the type C edge kl. (Recall Figure 5.5.)

<u>Subcase- C_1' has only one extra-skeletal edge</u>:

<u>$s > 2$</u>: Here G mimics one of MIN.18 or MIN.78, depending on whether kl is a normal or reversed type C edge in C_1'. (See Figures 6.39 and 6.40.) Note that even if $j_1 = a_1$ and $i_1 = b_2$, i_1j_1 contains the M formed by kl and b_2b_1, and is not u<u>seless</u>.

<u>$s = 2$</u>: With only two ij edges, we may use reversal and rotation to assume that kl is a normal type C edge, and $j_1 = j_2 = a_1$.

If $k < a_2$, then G mimics MIN.79. (See Figure 6.41.)

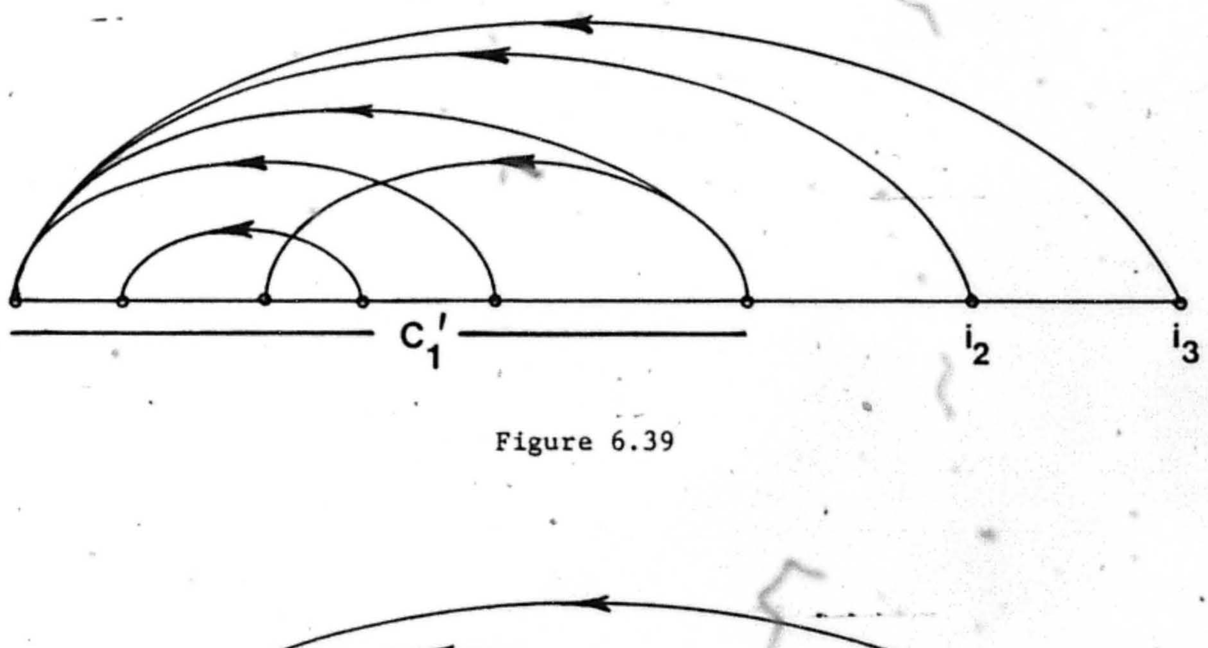


Figure 6.39

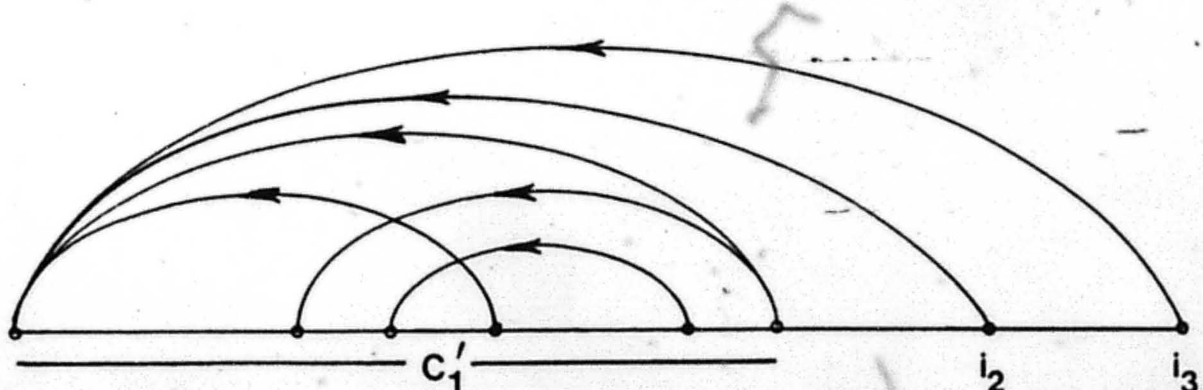


Figure 6.40

If $i_1 > b_2$, then G mimics MIN.18 (See Figure 6.42.)

If $k = a_2$ and $i_1 = b_2$ then a reduction of G is mimicked by MAX.21. (See Figure 6.43.)

<u>$s = 1$</u>: Here a reduction of G is mimicked by MAX.20. (See Figure 6.44.)

This concludes the case when C_1' has only one extra-skeletal edge.

<u>Subcase- C_1' has two extra-skeletal edges</u>: Let the extra-skeletal edges of C_1' be k_1l_1, k_2l_2. By the analysis of the previous chapter, we make the following case division.

<u>C_1' falls under C1(a)</u>: (Thus C_1' is as depicted in Figure 5.59.) Here G mimics MIN.80. (See Figure 6.45.)

<u>C_1' falls under C1(b)</u>: (Thus C_1' is as depicted in Figure 5.65.) Here G mimics MIN.81. (See Figure 6.46.)

<u>C_1' falls under C2(a)</u>: (Thus C_1' is as depicted in Figure 5.63.) Here G mimics MIN.82. (See Figure 6.47.)

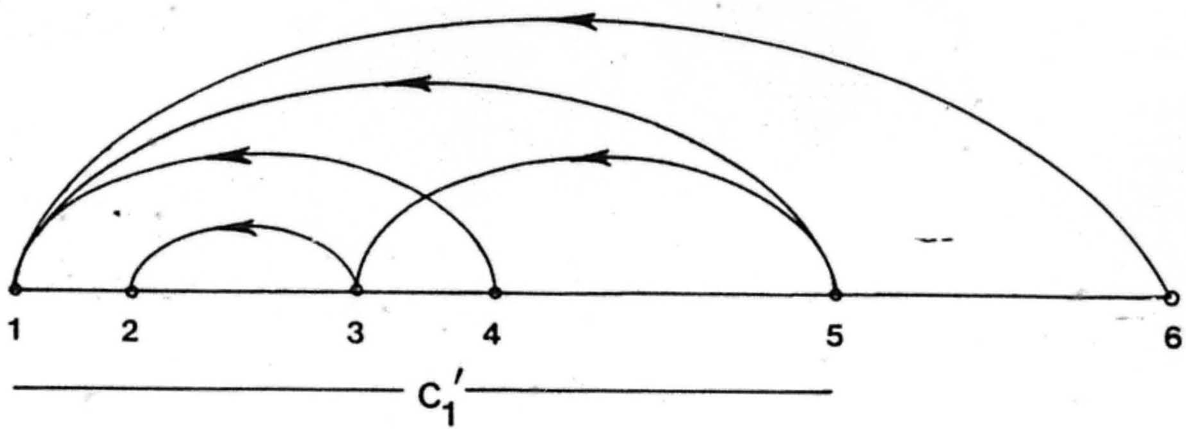


Figure 6.41

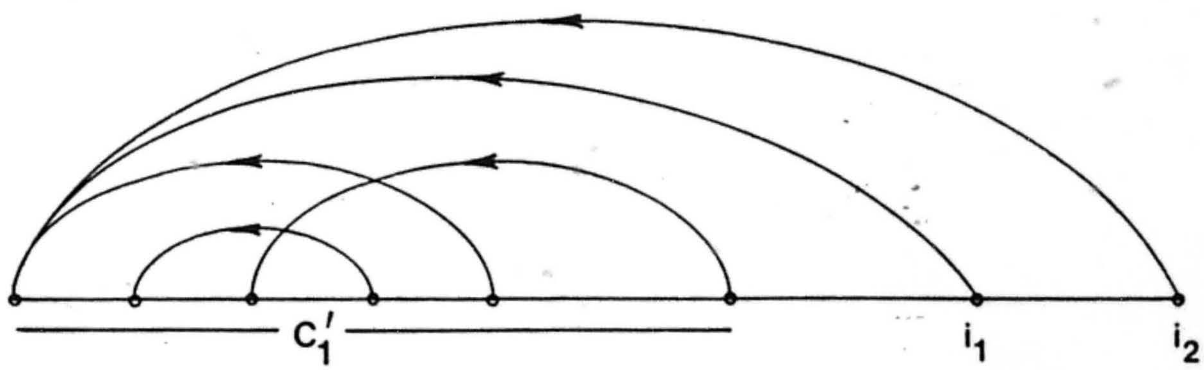


Figure 6.42

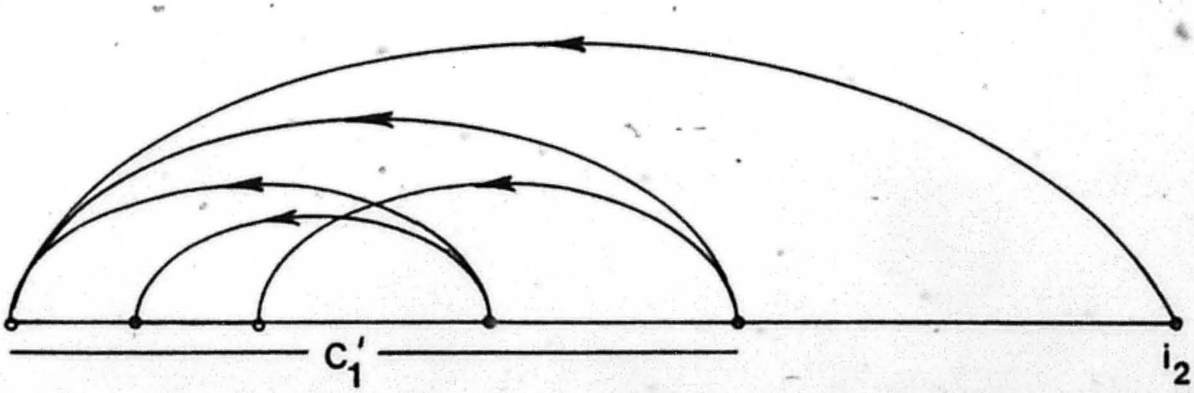


Figure 6.43

Figure 6.44

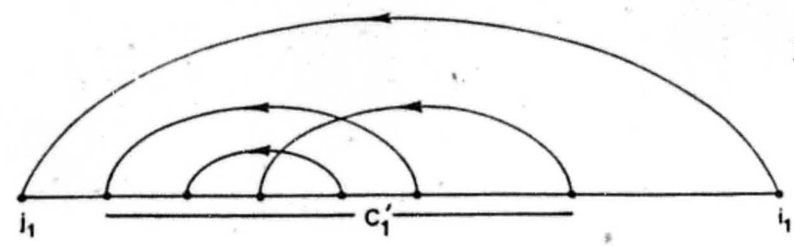


Figure 6.45

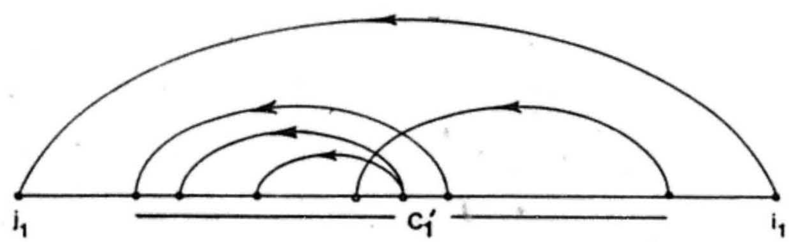


Figure 6.46

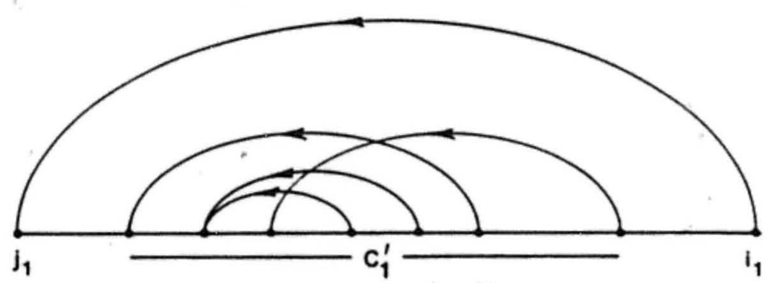


Figure 6.47

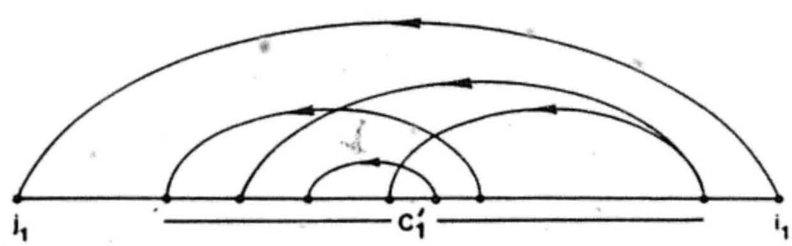
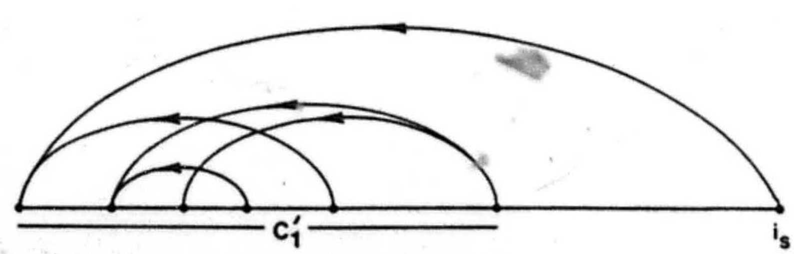


Figure 6.48



C$'_1$ falls under C2(b): (Thus C$'_1$ is as depicted in Figure 5.66.) We have two possibilities.

(i) $i_s \neq b_2$. Here G mimics MIN.83. (See Figure 6.48.)

(ii) $i_s = b_2$. By rotation we may assume that this means s = 1. Here a reduction of G is mimicked by MAX.19. (See Figure 6.49.)

C$'_1$ falls under C3(a): (Thus C$'_1$ is as depicted in Figure 5.73.) Here G mimics MIN.84. (See Figure 6.50.)

C$'_1$ falls under C3(b): (Thus C$'_1$ is as depicted in Figure 5.75.) We have two possibilities.

(i) $i_s = b_2$. By rotation we may assume that this means s = 1. Here a reduction of G is mimicked by MAX.18. (See Figure 6.51.)

(ii) $i_s \neq b_2$. Here G mimics MIN.85. (See Figure 6.52.)

C$'_1$ falls under C4: (Thus C$'_1$ is as depicted in Figure 5.57.) Here G mimics MIN.86. (See Figure 6.53.)

C$'_1$ falls under C5: (Thus C$'_1$ is as depicted in Figure 5.60.) We have two possibilities.

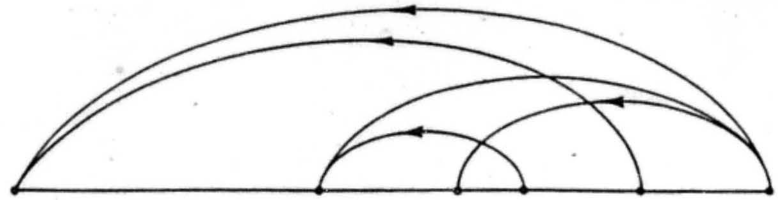


Figure 6.49

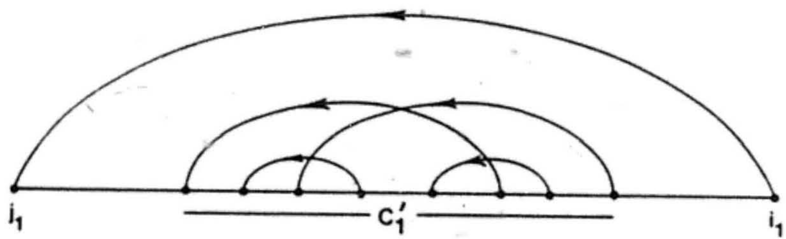


Figure 6.50

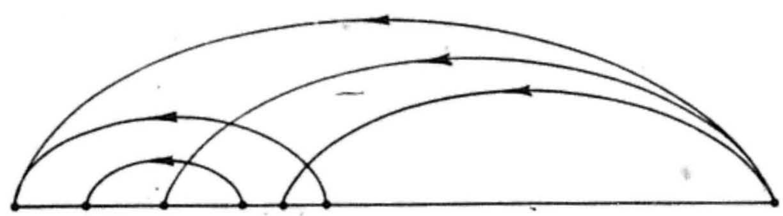


Figure 6.51

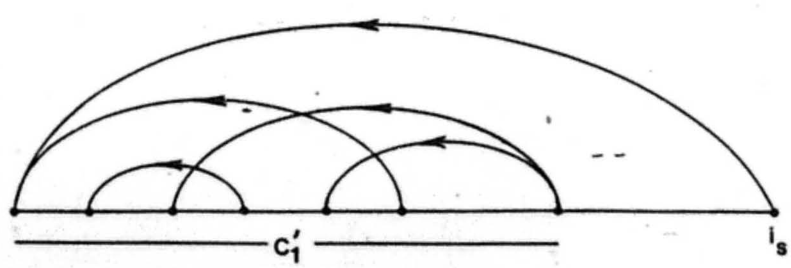


Figure 6.52

(i) $i_s \neq b_2$. Here G mimics MIN.87. (See Figure 6.54.)

(ii) $i_s = b_2$. By rotation we may assume that this means $s = 1$. Here a reduction of G is mimicked by MAX.17. (See Figure 6.55.)

Subcase- C_1' has three extra-skeletal edges:

We may again fall back on the classification of the previous chapter. Also, by the foregoing section, assume that $i_1 = b_2$ and that $s = 1$.

Subcase- C_1' falls under case C3(b) + C3(b): (Thus C_1' is as depicted in Figure 5.104.) Here G mimics MIN.88. (See Figure 6.56.)

Subcase- C_1' falls under case C3(b) + C2(b): (Thus C_1' is as depicted in Figure 5.107.) Here G mimics MIN.89. (See Figure 6.57.)

Subcase- C_1' falls under case C5 + C3(b): (Thus C_1' is as depicted in Figure 5.95.) Here G mimics MIN.49. (See Figure 6.58.)

This concludes our proof.

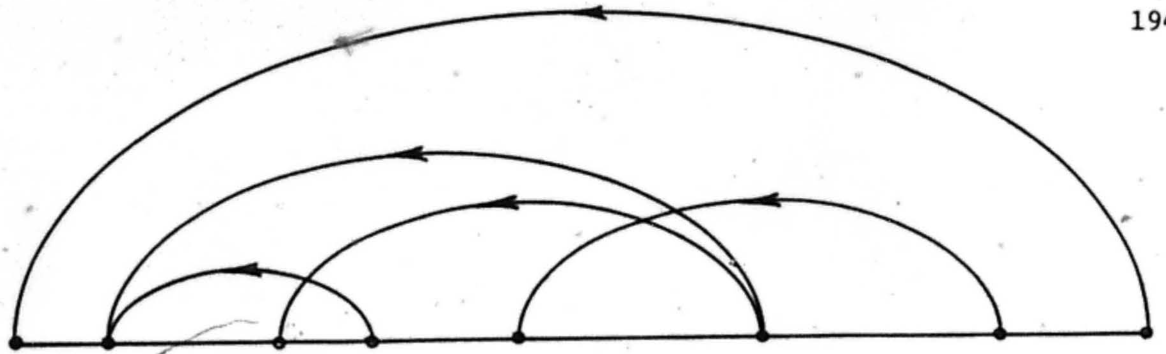


Figure 6.53

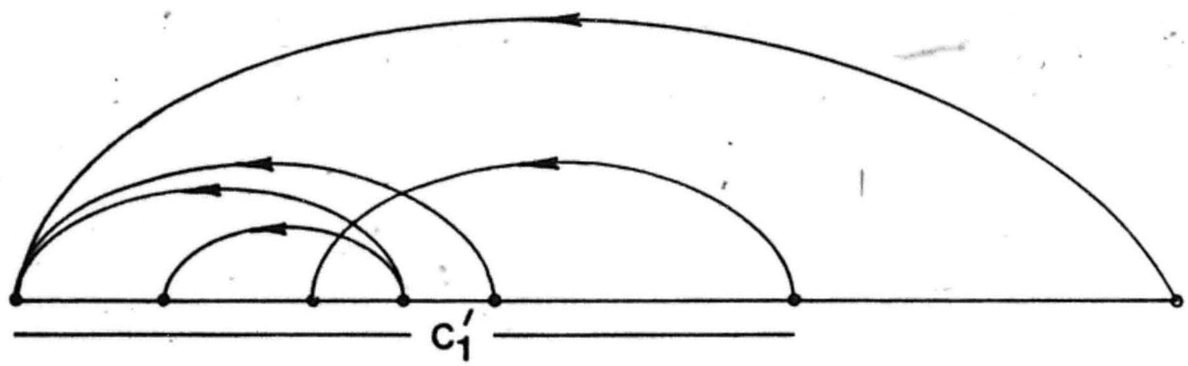


Figure 6.54

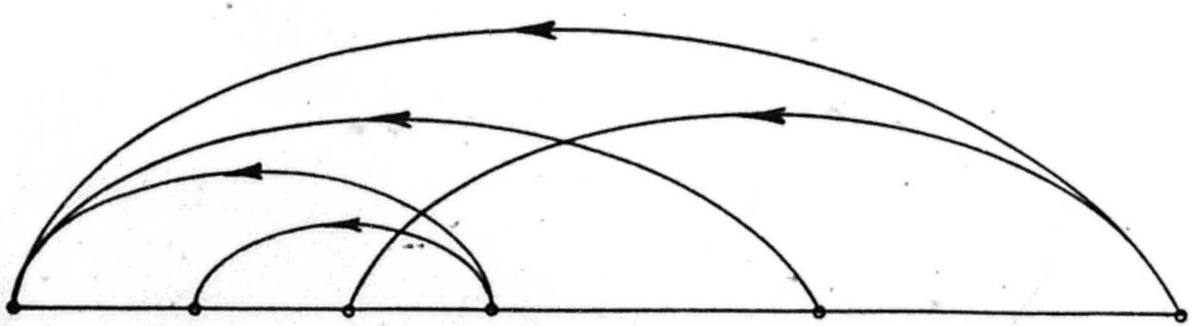


Figure 6.55

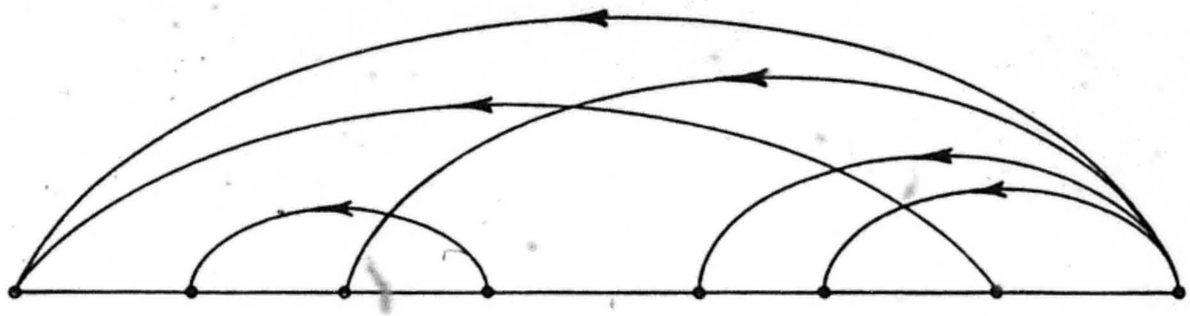


Figure 6.56

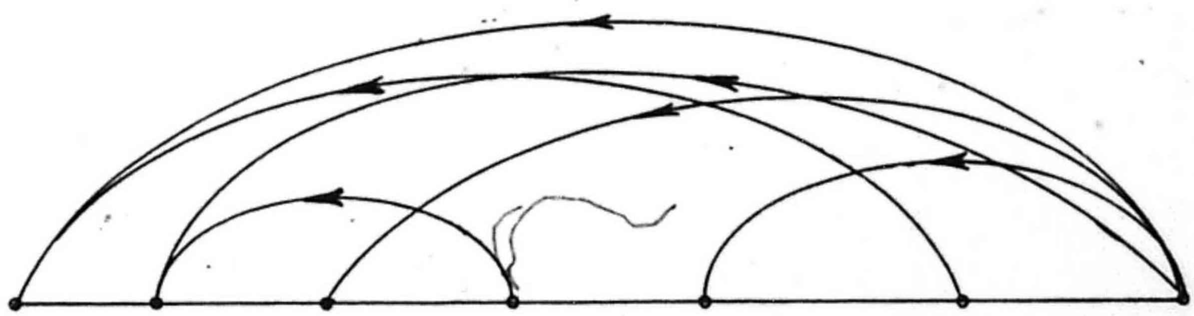


Figure 6.57

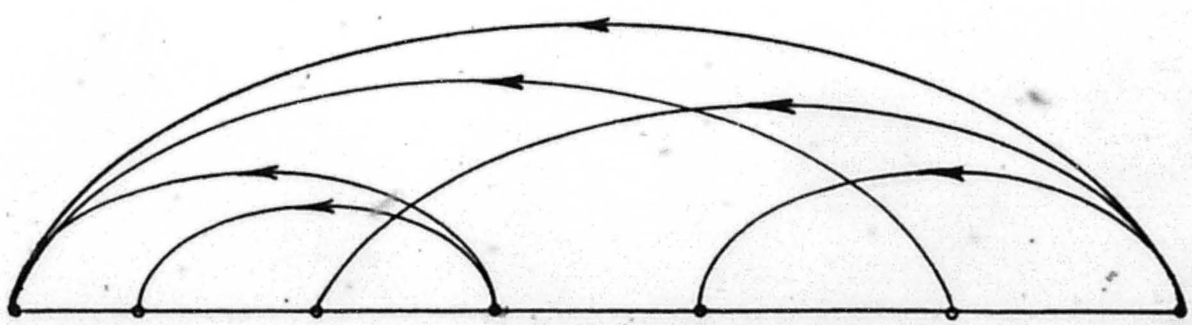


Figure 6.58

Chapter 7: Lemmas On Substitutions

Let $S = \{ x_1, x_2, x_3 \}$, T be alphabets. Let $g: S^* \to T^*$ be a substitution. We produce certain conditions on g which are sufficient to show that $g(h^\omega(x_2))$ is non-repetitive where h is substitution 2.1. That is,

$$h(x_1) = x_3$$
$$h(x_2) = x_2x_3x_1 \qquad (\text{Sub } 2.1)$$
$$h(x_3) = x_2x_1.$$

First we note that if the $g(x_i)$ each start in a distinctive way, but have sufficiently different endings, then g works.

Different Endings Lemma: Let A, B be alphabets, $T = A \cup B$, $A \cap B = \phi$. Suppose that for each i, we can write $g(x_i) = mb_i$, $m \in A$, $b_i \in B^*$ so that the following conditions hold:

(1) If for $1 \leq i, j, k \leq 3$ we can write $b_k = b_i'b_j''$ where $b_i = b_i'b_i''$, $b_j = b_j'b_j''$, then either $b_i' = \epsilon$, $j = k$, or $b_j'' = \epsilon$, $i = k$.. Thus we cannot glue together b_i from a prefix of b_j and a suffix of b_k.

(2) The word b_i is non-repetitive for each i.

Then $g(h^n(x_2))$ is non-repetitive for all n.

Proof: By Lemma 2.5, it suffices to show that the following conditions hold:

1') If $g(x)$ is a subword of $g(y)$, where $x, y \in S$, then $x = y$.

3') If $w \in S^*$ is a non-repetitive word, $|w| = 3$, then $g(w)$ is non-repetitive.

Clearly condition 1') holds: If mb_i is a subword of mb_j we must have b_i a prefix of b_j, whence $i = j$, by condition 1).

It remains to show that condition 3') is fulfilled. Suppose $g(x_i x_j x_k)$ contains a repetition for some i, j, k, $i \neq j$, $j \neq k$. Thus $mb_i mb_j mb_k$ contains a repetition vv, $v \neq \epsilon$. The word vv must contain exactly zero or two m's. If vv contains no m, then vv is a subword of b_j for some j, contradicting 2). On the other hand, if vv contains the first two m's of $mb_i mb_j mb_k$, then b_i is a prefix of b_j, impossible since $i \neq j$.

Finally, if vv contains the last two m's of $mb_i mb_j mb_k$, then we can write $v = b_i'' m b_j' = b_j'' m b_k'$, where $b_i = b_i' b_i''$, $b_j = b_j' b_j''$, $b_k = b_k' b_k''$. But then lining up the m's, we get $b_i'' = b_j''$, $b_j' = b_k'$, so that $b_j = b_k' b_i''$, contradicting 1). We conclude that $g(x_i x_j x_k)$ is non-repetitive whenever $i \neq j$, $j \neq k$, so that g fulfills condition 3'). □

Block/Separator Lemma: Suppose that we can write

$$g(x_i) = nb_0nb_{i+1}nb_i, \quad i = 1, 2, 3$$

where $b_i \in B^*$, each i, some alphabet B, $n \in A$, some alphabet A, such that $A \cap B = \phi$. Suppose further that the following conditions are fulfilled:

(1) If for $0 \leq i, j, k \leq 4$ we can write $b_k = b_i'b_j''$ where $b_i = b_i'b_i''$, $b_j = b_j'b_j''$, then either $b_i' = \epsilon$, $j = k$, or $b_j'' = \epsilon$, $i = k$..

(2) If $1 \leq i < j \leq 4$, then $|b_i| < |b_j|$.

(3) The word b_i is non-repetitive, $0 \leq i \leq 4$.

Then $g(h^n(x_2))$ is non-repetitive for all n.

Proof: Our proof is analogous to the previous proof, but somewhat more involved. Again condition 1') of Lemma 2.5 will hold. It remains to show that condition 3') holds.

Suppose $g(x_ix_jx_k)$ contains a repetition for some i,j,k, $i \neq j$, $j \neq k$. Then we get

$nb_0nb_pnb_q\ nb_0nb_rnb_s\ nb_0nb_tnb_u$ containing a repetition vv, $v \neq \epsilon$.

Case A: The word v contains nb_0n as a subword. Examining g, we see that vv must contain this subword exactly twice. If vv contains the first two occurrences of this subword in $g(x_ix_jx_k)$ then b_p is a prefix of b_r. By condition (1), $p = r$, so that $i = j$, a contradiction.

Thus it must be the second two occurrences of nb_0n

which are in vv, and lining things up using these matching subwords in vv, we write

$$(b_p n b_q)'' = (b_r n b_s)'', \quad (b_r n b_s)' = (b_t n b_u)',$$

where as usual x' (x") stands for a prefix (suffix) of x, and $b_r n b_s = (b_r n b_s)'(b_r n b_s)''$. But if n is in $(b_r n b_s)''$, then $b_s = b_q$, so that s = q, and i = j, a contradiction. However, then n must be in $(b_r n b_s)'$, whence $b_r = b_t$, again giving a contradiction. We conclude that nb_0n is not a subword of v.

<u>Case B</u>: The word vv contains a subword nb_0n, but v does not.

Thus nb_0n "straddles the border" between the two v's of vv, and we write $v = Znb_0' = b_0''nY$, where $b_0'b_0'' = b_0$, $Z, Y \in (A \cup B)^*$. Thus $w = g(x_i x_j x_k)$ contains a subword of the form $b_0''nXnb_0nXb_0'$, $X \in (A \cup B)^*$. However, now by condition (1), we may assume that w contains either nb_0nXnb_0nX or Xnb_0nXnb_0n as a subword, and we are back in Case A, which has already been dealt with.

<u>Case C</u>: The repetition vv does not contain nb_0n. Thus without loss of generality, up to reindexing, assume that vv is a subword of $b_0nb_rnb_snb_0$.

By condition (3), vv contains at least one n. Thus vv contains exactly two n's. If the first two n's here are in vv, then b_r is a prefix of b_s, contradicting (1).

If the second two n's are in vv, then we write $b_s = b_0'b_r''$ and get the usual contradiction.

We conclude that $g(x_i x_j x_k)$ is non-repetitive.□

Long/Short Lemma: Suppose that we can write

$$g(x_1) = mb_1e_1$$

$$g(x_2) = mb_2e_1b_1e_2$$

$$g(x_3) = mb_2e_2$$

where for each i, b_i, $e_i \in A^*$, some alphabet A, $m \in B$, some alphabet B, such that $A \cap B = \phi$. Then if

(1) Each of $g(x_1)$, $g(x_2)$, $g(x_3)$ is non-repetitive

(2) $| b_1 | < | b_2 |$, $| e_1 | < | e_2 |$

(3) If w is a common prefix of b_2, b_1, v a common suffix of e_1, e_2, then $| wv | < | b_1e_1 |$

(4) Any common prefix of b_1e_i, b_2e_j is of length $\leq | b_1 |$

(5) Any common suffix of b_ie_1, b_je_2 is of length $\leq | e_1 |$

then g is suitable.

Proof: We show that g is suitable, i.e.

1) $| g(x_i) | \leq | g(x_j) | + | g(x_k) |$ for $1 \leq i, j, k \leq 3$, i, j, k distinct

2) For $1 \leq i \leq 3$, one cannot write $g(x_i) = uw = wz$, u, w, z non-empty words over T.

(3) If $w \in S^*$ is a non-repetitive word with $| w | \leq 3$, and $w \neq x_1x_3x_1$, $x_2x_3x_2$, then g(w) is non-repetitive.

Conditions 1) and 2) are easily checked. It remains to show that condition 3) holds. It will be useful to first consider the case when $|w| = 2$.

$\underline{w = x_1 x_2:}$ Here $g(w) = mb_1e_1mb_2e_1mb_1e_2$. Suppose that vv is a subword of g(w) for some $v \neq \epsilon$. By condition (1), repetition v must include the second m of g(w); any other repetition would be entirely inside $g(x_1)$ or $g(x_2)$. There are two possibilities:

Case 1: The first two m's are in vv. In this case we must have m for a prefix of v, so that $v = mb_1e_1$. Therefore, $mb_1e_1 = v$ is a prefix of mb_2e_1. Thus b_1e_1 is a prefix of b_2e_1. This is a contradiction of condition (4), as $| b_1e_1 | > | b_1 |$. Thus g(w) is non-repetitive in this case.

Case 2: The second two m's are in vv. Thus vv is contained in the word $b_1e_1mb_2e_1mb_1e_2$. Using the m's to line up the pieces v, we have $(b_1e_1)'' = (b_2e_1)''$, $(b_2e_1)' = (b_1e_2)'$ where $(b_ie_j)'$ stands for a non-empty prefix of b_ie_j, $(b_ie_j)''$ stands for a non-empty suffix of b_ie_j, and $(b_2e_1)'(b_2e_1)'' = b_2e_1$. However, by condition (4), $| (b_2e_1)' | \leq | b_1 |$. Lining up the e_1's, we can therefore write $b_2 = b_1'b_1''$, where $b_1 = xb_1'' = b_1'y$ for some x,y. Since $| b_2| > |b_1|$, we can write $b_1'' = zy$, b_1'

= xz for some $z \neq \epsilon$. But then b_2 contains the repetition zz, a contradiction. (We call this an overlap argument.). Thus g(w) can have no repetition.

$\underline{w = x_1x_3:}$ In this case $g(w) = mb_1e_1mb_2e_2$. Any repetition vv involves both m's, and lining things up using the m's, we find that b_1e_1 is a prefix of b_2e_2, contradicting condition (4).

$\underline{w = x_2x_1:}$ Here $g(w) = mb_2e_1mb_1e_2mb_1e_1$. Any repetition vv must involve the second two m's, as the first two are contained in $g(x_2)$. Then we get $(b_2e_1)'' = (b_1e_2)''$, $(b_1e_2)' = (b_1e_1)'$, and $b_1e_2 = (b_1e_2)'(b_1e_2)''$. Now by condition (5), $(b_1e_2)''$ must actually be a suffix of e_1 alone. Lining up b_1's, we get $e_2 = e_1'e_1''$ and we use an overlap argument as in a previous case.

$\underline{w = x_2x_3:}$ We get $g(w) = mb_2e_1mb_1e_2mb_2e_2$. Any repetition involves the last two m's. We argue similarly to the previous case, except now we get $|(b_1e_2)'| \leq |b_1|$, $|(b_1e_2)''| \leq |e_1|$. This forces $|b_1e_2| \leq |b_1e_1|$, a contradiction.

$\underline{w = x_3x_1:}$ Here $g(w) = mb_2e_2mb_1e_1$. If there were a repetition, we would have mb_2e_2 a prefix of mb_1e_1, which is absurd because of the respective lengths.

<u>$w = x_3x_2$:</u> We have $g(w) = mb_2e_2mb_2e_1mb_1e_2$. Any repetition must match the first two m's, forcing b_2e_2 to be a prefix of mb_2e_1, which is absurd.

We have thus established that g behaves well on the two letter words. It remains to consider the cases when $|w| = 3$:

<u>$w = x_1x_2x_1$:</u> Here $g(w) = mb_1e_1mb_2e_1mb_1e_2mb_1e_1$. Since the g behaves well on two letter words, any repetition vv in g(w) must straddle the images of all three letters here, thus containing at least the last three m's. We conclude that (since repetitions contain an even number of m's) all four m's are in vv. This implies that $mb_1e_1 = mb_1e_2$, which is absurd, as the lengths differ.

<u>$w = x_1x_2x_3$:</u> Here $g(w) = mb_1e_1mb_2e_1mb_1e_2mb_2e_2$, and we get the same contradiction as in the previous case. This contradiction will similarly occur for each w of form $x_ix_2x_j$.

<u>$w = x_1x_3x_2$:</u>

Here $g(w) = mb_1e_1mb_2e_2mb_2e_1mb_1e_2$. Here either all four m's get matched up, or only the central two. If all four m's get matched, we get an absurdity involving lengths. We therefore assume that the second two m's are matched, and write $(b_1e_1)'' = (b_2e_2)''$, $(b_2e_2)' = (b_2e_1)'$ where $(b_ie_j)'$ stands for a non-empty prefix of b_ie_j, $(b_ie_j)''$ stands for a non-empty suffix of b_ie_j, and $(b_2e_2)'(b_2e_2)'' = b_2e_2$. However, by condition (5), $|(b_2e_2)''| \leq |e_1|$. Lining up the b_2's we get a contradiction by an overlap argument.

<u>$w = x_2x_1x_2$:</u> Here $g(w) = mb_2e_1mb_1e_2mb_1e_1mb_2e_1mb_1e_2$ We match either the third and fourth m's, the first four m's, or the last four m's.

If the third and fourth m's are matched, after our usual argument we end up with b_1 a prefix of b_2, e_1 a suffix of e_2. This is forbidden by condition (3).

Matching the first four m's gives our standard absurdity involving length. Thus suppose the last four m's are matched. This forces $b_1e_2 = b_2e_1$, contradicting the non-repetitiveness of $g(x_2) = mb_2e_1mb_1e_2$.

<u>$w = x_2x_1x_3$:</u> Here $g(w) = mb_2e_1mb_1e_2mb_1e_1mb_2e_2$. We cannot have all four m's in a repetition, as this gives the usual contradiction concerning lengths. The alternative is that the last two m's are matched by a repetition and here we get the same contradiction as in the previous

case.

<u>$w = x_2x_3x_1$</u>: Here $g(w) = mb_2e_1mb_1e_2mb_2e_2mb_1e_1$. Involving all four m's in a repetition is impossible, as usual. However, matching the last two m's gives a contradiction by the overlap argument.

<u>$w = x_3x_1x_2$</u>: Here $g(w) = mb_2e_2mb_1e_1mb_2e_1mb_1e_2$. Here either all four m's get matched up, or only the middle two. If all four m's get matched, we get an absurdity involving lengths. If the center two m's are matched, then after our usual argument we end up with b_1 a prefix of b_2, e_1 a suffix of e_2. This is forbidden by condition (3).

<u>$w = x_3x_1x_3$:</u> Here $g(w) = mb_2e_2mb_1e_1mb_2e_2$. The first two m's cannot be paired. However matching the last two we end up with b_1 a prefix of b_2, e_1 a suffix of e_2. This is forbidden by condition (3).

$mb_1e_1 = mb_1e_2$, which is absurd, as the lengths differ.

<u>$w = x_3x_2x_1$:</u> As we remarked earlier the contradiction of the cases $w = x_1x_2x_1$, $x_1x_2x_3$ carries over to this case and the next.

<u>$w = x_3x_2x_3$</u>: See above.

Having looked at all the short words and finding g to be well-behaved, we are finished our proof.□

<u>MIN.71:</u> To deal with MIN.71, we use some substitutions on a five letter alphabet. Let $l:\{1, 2, 3, 4, 5\} \to \{1, 2, 3, 4\}$ be given by

$l(1) = 43$

$l(2) = 4321$

$l(3) = 432153$

$l(4) = 41521$

$l(5) = 4153$

Let $k: \{1, 2, 3, 4, 5\} \to \{1, 2, 3, 4, 5\}$ be given by

$k(1) = 51234321234512321234 3234$

$k(2) = 51234321234512321234323451232 1234$

$k(3) = 5123432123451232123432 34$-

$5123212345123432345123212343234$

$k(4) = 5123432123451234323451232 1234$

$k(5) = 5123432123451234323451232123432 34$

We wish to show that $k(l^{\omega}(4))$ is non-repetitive. Consider the following simplified substitution lemma.

<u>Lemma:</u> Let $g:A^* \to B^*$ be a substitution so that

(1) If $g(a_1 \ldots a_n) = X\, g(e)\, Y$, then for some j, $X = g(a_1 \ldots a_j)$, $a_{j+1} = e$.

(2) If we can write $g(x_i)'' = g(x_j)''$, $g(x_j)' = g(x_k)'$, $g(x_j) = g(x_j)'g(x_j)''$ with x' (x'') a prefix (suffix) of x, then $w = x_i x_j x_k$ is among $w_1, w_2, \ldots, w_n$.

Then if v is a non-repetitive word never containing $x_i X x_j X x_k$ where $x_i x_j x_k = w_r$, some r, $1 \leq r \leq n$, any word X, then $g(v)$ is non-repetitive.w_3,$g(x_i x_j x_k) = | w | = 3$, then $g(w)$ is non-repetitive unless $w = w_1, w_2, \ldots, w_m$.

This result follows from the proof of the substitution lemma. The " line-up " condition (1), will clearly be true of k and l. The following list may be checked to suffice for condition (2) for substitutions k and l: 123, 145, 154, 212, 213, 241, 242, 243, 245, 312, 313, 314, 315, 323, 345, 351, 352, 353, 354, 412, 413, 421, 423, 512, 513, 514, 515, 523, 532, 545

Suppose that $x_i X x_j X x_k$ never appears in $l^m(4)$ for any m. Then $l^m(4)$ is non-repetitive for each m, by the simplified substitution lemma, and so is $k(l^\omega(4))$. Thus to show that $k(l^\omega(4))$ is non-repetitive it suffices to show that none of the listed triples occurs in this way in $l^n(4)$. We now do this:

Suppose that for some m, one of the listed triples occurs in the above manner in $l^m(4)$. Choose m as small as possible. In particular, $1^{m-1}(4)$ contains none of the above listed triples, so that $1^m(4)$ is non-repetitive.

Triple 123: Suppose that there exists a word $X = a_1a_2..a_r$ so that 1X2X3 appears in $1^m(4)$. Then a_1 follows 2 in $1^m(4)$, so by examining 1, we conclude that a_1 is 1. However, then $1^m(4)$ contains 11, which, again examining 1, we see is impossible.

Triple 254: Suppose that there exists a word $X = a_1a_2..a_r$ and a letter y, so that yX5X4 appears in $1^m(4)$. Then a_1 follows 2, and must be a 1. Then $1^m(4)$ contains 51, which is impossible.

Triples 212, 312, 412, 512: Suppose that there exists a word $X = a_1a_2..a_r$ and a letter y, so that yX1X2 appears in $1^m(4)$. Then a_r precedes 1, and must be a 2 or a 4. However a_r precedes 2, and so must be a 3 or a 5. This is a contradiction.

Triples 213, 313, 413, 513: Suppose that there exists a word $X = a_1a_2..a_r$ and a letter y, y = 2, 3, 4, or 5, so that yX1X3 appears in $1^m(4)$. Then a_r is a 4, because it precedes both 3 and 1 in pieces from 1. Then a_1 follows 41, and must be a 5. However, y precedes a_1 and thus must be a 1, contradicting our choice of y.

Triple 314: Suppose that there exists a word $X = a_1a_2..a_r$ so that 3X1X4 appears in $1^m(4)$. Then a_r precedes 4, and is a 1 or 3. However neither 31 nor 11 can appear, so we have a contradiction.

Triple 315: Suppose that there exists a word $X = a_1a_2..a_r$ so that 3X1X5 appears in $1^m(4)$. Then a_r precedes 5, and must be a 1. Then $1^m(4)$ contains 11, which is impossible.

Triples 323: Suppose that there exists a word $X = a_1a_2..a_r$ so that 3X2X3 appears in $1^m(4)$. Then a_1 follows 2 and must be a 1. But then $1^m(4)$ contains 31, which is impossible.

Triples 351, 352, 353: Suppose that there exists a word $X = a_1a_2..a_r$ and a letter y, y = 1, 2 or 3, so that 3X5Xy appears in $1^m(4)$. Then a_r precedes 5, and must be a 1. This forces y to be a 4 or a 5, contradicting our choice of y.

Triple 421: Suppose that there exists a word $X = a_1a_2..a_r$ so that 4X2X1 appears in $1^m(4)$. Then a_r precedes 2, and is a 3 or a 5. However, as a_r precedes 1, it must be a 4 or a 2. This is a contradiction.

Triple 423: Suppose that there exists a word $X = a_1a_2..a_r$ so that 4X2X3 appears in $1^m(4)$. Then a_1 precedes 2, and must be a 1. Now a_2 is preceded by 41, and must be a 5. Thus a_r is followed by 215, and is a 3. But then $1^m(4)$ contains 33, which is a contradiction.

Triples 513, 514, 515: Suppose that there exists a word

$X = a_1a_2..a_r$ and a letter y, so that 5X1Xy appears in $1^m(4)$. Then a_1 follows 1 and must be 5 or 4. However, a_1 follows 5, and must be 3 or 2. This is a contradiction.

<u>Triples 523</u>: Suppose that there exists a word $X = a_1a_2..a_r$ so that 5X2X3 appears in $1^m(4)$. Then a_1 follows 2 and must be 1, leaving $1^m(4)$ to contain 51, which is impossible.

<u>Triple 532:</u> Suppose that there exists a word $X = a_1a_2..a_r$ so that 5X3X2 appears in $1^m(4)$. Then a_1 follows 5 and 3, and is forced to be a 2. Thus a_r must be a 4, as it is followed by 32. But then $1^m(4)$ contains 42, which is impossible.

<u>Triples 145, 245, 345, 545:</u> Suppose that there exists a word $X = a_1a_2..a_r$ and a letter y, so that yX4X5 appears in $1^m(4)$. Then a_r precedes 5, and must be a 1. If r = 1, then $1^m(4)$ contains 141, so that $1^{m-1}(4)$ contains one of 24, 25, 44, 45, which is impossible, since 1 never produces these words. Thus r > 1. Since a_1 follows 4, a_1 is 3 or 1. However a_1 cannot be 1, or $1^m(4)$ contains 141.

Now y precedes 3, and must be 4 or 5. However, if y = 4, then $1^m(4)$ contains the repetiton 4X4X, contradicting the minimality of m. Thus y = 5. Since a_2 follows 53, a_2 = 4.

Continuing these kinds of arguments, it may be shown that in fact $1^m(4)$ contains a block of the form

$$53\ 41521\ Z\ 1^2(4)\ 41521\ Z\ 1^2(5).$$

However the block 41521 Z is of the form $1^2(Q)$. We thus deduce that $1^{m-2}(4)$ contains Q4 Q5. This contradicts the minimality of m.

Triples 241, 242, 243: Suppose that there exists a word $X = a_1a_2..a_r$ and a letter y, $y \neq 4, 5$, so that 2X4Xy appears in $1^m(4)$. Then a_1 must be a 1 as it follows 2 and 4. Then a_2 follows 41, and is a 5, a_3 follows 215, and is a 3. Thus a_3 follows 2153, and is a 4.

As a_r is followed by a 4, a_r is 3 or 1. But a_r is not 1, otherwise $1^{m-1}(4)$ would contain one of 24, 25, 44, 45, which is impossible. Thus a_r is a 3. Then y is a 4 or 2. However, y cannot be a 4, or $1^m(4)$ contains the repetition X4X4, which contradicts the minimality of m.

Continuing these kinds of arguments, it may be shown that in fact $1^m(4)$ contains a block of the form

$$1^2(2)\ 41521\ Z\ 1^2(4)\ 41521\ Z\ 41521432.$$

However the block 41521 Z is of the form $1^2(Q)$. We thus deduce that $1^{m-2}(4)$ contains 2Q 4Q. This contradicts the minimality of m.

Since none of the bad triples occur, $k(1^m(4))$ is non-repetitive for every m.□

Chapter 8: Substitutions

We wish to show that each of the graphs of MIN is versatile. We saw at the end of Chapter 2 that MIN.1 - MIN.4 are versatile. MIN.71 is treated separately in Chapter 7. For each other digraph G of MIN, we give a substitution f meeting the demands of the Substitution Lemma (Lemma 2.4), such that $f(g^{\omega}(2))$ is a non-repetitive walk of type ω on G, where g is substitution 2.1.

The conditions of the Substitution Lemma are such that they are easily verified for each of these substitutions by computer. However, to aid the understanding of the reader, we label substitutions with the labels 'D/E', 'B/S', 'L/S' standing for 'Different/Endings', 'Block/Separator' and 'Long/Short' respectively. A substitution with such a label falls into the corresponding category of substitutions as discussed at the end of Chapter 3, and can usually be shown to be suitable using the corresponding theorem of Chapter 7.

MIN.5: x_1: 12342
x_2: 1234542123423 L/S
x_3: 12345423

MIN.6: x_1: 1234543
x_2: 12343234543123454323 43 L/S
x_3: 123432345432343

MIN.7: x_1: 12345423
x_2: 123454342 3[illegible]234542343 L/S
x_3: 12345434 2343

MIN.8: x_1: 1234523
x_2: 12345652312345 2343 L/S
x_3: 12345652343

MIN.9: x_1: 1343
x_2: 12343134543 L/S
x_3: 1234543

MIN.10: x_1: 13423
x_2: 1234231345423
x_3: 12345423

MIN.11: x_1: 1434543
x_2: 1234314543
x_3: 1231431234543

MIN.12: x_1: 14542
x_2: 1234542142342
x_3: 1234542342

MIN.13: x_1: 12343
x_2: 164345643
x_3: 1645643

MIN.14: x_1: 123454

x_2: 123212345412345654 L/S

x_3: 123212345654

MIN.15: x_1: 12345234

x_2: 1234323452341234523 43234 L/S

x_3: 123432345234 3234

MIN.16: x_1: 123454

x_2: 123432345412345 43234 L/S

x_3: 12343234543234

MIN.17: x_1: 1234565452

x_2: 123456523452123456545232 L/S

x_3: 12345652345232

MIN.18: x_1: 12345

x_2: 123432345123456 L/S

x_3: 1234323456

MIN.19: x_1: 12345674

x_2: 123212345674123456765674 L/S

x_3: 123212345676 5674

MIN.20: x_1: 12321234567

x_2: 123456545671232123456787 L/S

x_3: 1234565456787

MIN.21: x_1: 123212345

x_2: 123432345123212 34565 L/S

x_3: 12343234565

MIN.22:	x_1:	123534	
	x_2:	123453412353423 4	L/S
	x_3:	1234534234	
MIN.23:	x_1:	1234212345	
	x_2:	12342312345123421234565	L/S
	x_3:	1234231234565	
MIN.24:	x_1:	123432345	
	x_2:	12343123451234323 4565	L/S
	x_3:	123431234565	
MIN.25:	x_1:	12321234567612345 6	
	x_2:	12321234561234 5	B/S
	x_3:	1232123451234	
MIN.26:	x_1:	1232123451234	
	x_2:	1232123456412345	
	x_3:	123212345645	
MIN.27:	x_1:	1234565	
	x_2:	12321234565123456 45	L/S
	x_3:	123212345645	
MIN.28:	x_1:	123453	
	x_2:	12345231234534	L/S
	x_3:	12345234	
MIN.29:	x_1:	12345621234565	L/S
	x_2:	12345623123456512345621234564 5	
	x_3:	1234562312345645	

MIN.30: x_1: adacb

x_2: adc

x_3: aeb

where a = 1234565

where b =1234562

where c =12345623

where d =123456234

where e =1234562345

MIN.31: x_1:12342

x_2:12345342123423 L/S

x_3:123453423

MIN.32: x_1: 12342312345342345323123453423123453

x_2: 123423123453423453123453234231234534234534

x_3: 12342312345342345323423123453

MIN.33: x_1: 12342312345342341234534 B/S

x_2: 1234231234534234531234534234

x_3: 12342312345341234534

MIN.34: x_1: 1234534

x_2: 123212345341234532344 L/S

x_3: 1232123453234

MIN.35: x_1: 1234231234534234534 L/S

x_2: 12342312345342345123453123423-

12345342345312345

	x_3:	12342312345342345123453l2345	
MIN.36:	x_1:	12345634	
	x_2:	1234563234123456345	L/S
	x_3:	12345632345	
MIN.37:	x_1:	1234564345234564 5	L/S
	x_2:	12345645234564345234564512345643452345645234 5	
	x_3:	123456452345643452345645234 5	
MIN.38:	x_1:	123454234	
	x_2:	12345342341234542345 34	
	x_3:	123453423454	
MIN.39:	x_1:	12345434	
	x_2:	12345423412345434 2	L/S
	x_3:	1234542342	
MIN.40:	x_1:	1234567345	
	x_2:	1234567323451234567345 65	L/S
	x_3:	12345673234565	
MIN.41:	x_1:	2123454345	
	x_2:	2341234543452123454123 45	
	x_3:	2341234543412345	
MIN.42:	x_1:	1232123456765612345676	
	x_2:	1232123456761234 56	B/S
	x_3:	12321234561234	
MIN.43:	x_1:	1232123454	
	x_2:	1234212345412321234542 34	

x_3: 12342123454234

MIN.44: x_1: 1234564541234563234 L/S

x_2: 123456345412345632341234564541234563234564

x_3: 123456345412345632345 64

MIN.45: x_1: 12345631234565

x_2: 1234563231234563 45

x_3: 12345632345

MIN.46: x_1: 123456434

x_2: 1234564234123456 4345 L/S

x_3: 12345642345

MIN.47: x_1: 12342

x_2: 123432123423 L/S

x_3: 1234323

MIN.48: x_1: 12345623451234564345623456434 5-

1234564345623456 5 B/S

x_2: 12345623451234564345623456512345643456 5

x_3: 1234562345123456434565123456434 5

MIN.49: x_1: 12345634

x_2: 1234562341234563454 L/S

x_3: 12345623454

MIN.50: x_1: 1234564345

x_2: 12345642345123456434565 L/S

x_3: 1234564234565

MIN.51: x_1: 12345632345

x_2:123456342345123456323456 5 L/S

x_3:12345634234565

MIN.52: x_1: 123452341234543452345434123454345 34

x_2: 12345234123454345234541234534

x_3: 123452341234543452345341234543 4

MIN.53: x_1: 1234543423454

x_2: 1234543452345412345434234542 34

x_3: 123454345234543 4

MIN.54: x_1: 12345653452345654 5

x_2: 12345654534565234 5 D/E

x_3: 12345654523456534 5

MIN.55: x_1: 1234534

x_2: 123432345341234532 34 L/S

x_3: 1234323453234

MIN.56: x_1: 123454

x_2: 123212345412345 64 L/S

x_3: 12321234564

MIN.57: x_1: 12345673234567612345674 56

x_2: 1234567323456756123456745 6-

1234567323456761234567345 6 L/S

x_3: 12345673234567561234567345 6

MIN.58: x_1: 1234564512345632345623451234563234565

x_2: 1234563451234563234562345123456323456 5-

1234564512345632345623451234563234564 5-

12345623451234563234565 L/S

x_3: 123456345123456323456234512345632345645-

12345623451234563234565

MIN.59: x_1: 123456342345632345

x_2: 1234563452345634234562345 D/E

x_3: 12345634523456323456 2345

MIN.60: x_1: 2345612345673456234567345

x_2: 23456123456734523456734 B/S

x_3: 2345612345673423456 73

MIN.61: x_1:1232123456789

x_2:1234565456789123212345678987 89 L/S

x_3:1234565456789 8789

MIN.62: x_1: 12321234567

x_2: 1234321234567123212345676567 L/S

x_3: 12343212345676567

MIN.63: x_1: 23456781

x_2: 2343234567812345678767 81 L/S

x_3: 2343234567876781

MIN.64: x_1: 123421234567 L/S

x_2: 1234231234567123421234567 6567

x_3: 12342312345676567

MIN.65: x_1: 123431234567 L/S

x_2: 12343231234567123431234567 6567

x_3: 123432312345676567

MIN.66: x_1: 1232

x_2: 123454 D/E

x_3: 123456

MIN.67: x_1: 1234567876785678 L/S

x_2: 123212345678767856781234567876784567 8

x_3: 12321234567876784567 8

MIN.68: x_1: 12345678

x_2: 1232123456787678

x_3: 1234123212345678767 85678

MIN.69: x_1: 123456

x_2: 12321234565456123456 7

x_3: 123212345654567

MIN.70: x_1: 123212345

x_2: 1234212345123212345423 45 L/S

x_3: 123421234542345

MIN.72: x_1: 1232123451234

x_2: 123212345612345 B/S

x_3: 1232123456712 3456

MIN.73: x_1: 1232123456512345 6

x_2: 123212345612345

x_3: 1232123451234

MIN.74: x_1: 12345671234562345123456234

x_2: 1234567123456234123456 23 B/S

x_3: 1234567123456231234562

MIN.75: x_1: 1234212345

x_2: 1234231234512342123456 L/S

x_3: 123423123456

MIN.76: x_1: 12342

x_2: 12343 D/E

x_3: 12345

MIN.77: x_1: 123453

x_2: 12343234531234532343 L/S

x_3: 12343234532343

MIN.78: x_1: 1234212345

x_2: 12342321234512342123456 L/S

x_3: 1234232123456

MIN.79: x_1: 1234534

x_2: 123456 D/E

x_3: 1234532345

MIN.80: x_1: 12345434

x_2: 123454234123454345 L/S

x_3: 1234542345

MIN.81: x_1: 123453234

x_2: 12345342341234532345 L/S

x_3: 12345342345

MIN.82: x_1: aec

x_2: afcaecb L/S

x_3: afcb

where a = 12345234

b = 12345434

c = 123454345

e = 12345434523454

f = 123454345234 5434

MIN.83: x_1: 12345

x_2: 123432345123423 L/S

x_3: 1234323423

MIN.84: x_1: 123456123454

x_2: 1234563234123456 3454

x_3: 123456323456123456323454

MIN.85: x_1: 123454

x_2: 12345323454123456 L/S

x_3: 12345323456

MIN.86: x_1: 1232123432341234323

x_2: 1232123432312343 B/S

x_3: 1232123431234

MIN.87: x_1: 123456

x_2: 1234534 D/E

x_3: 12345323

MIN.88: x_1: 12345632345612345645 L/S

x_2: 123456323456512345645123456323456123456345

x_3: 12345632345651234 56345

MIN.89: x_1: afd

x_2: bfdafecd L/S

x_3: bfecd

where a = 123454

b = 1234534

c = 12345234

d = 1234532345

e = 12345323454

f = 1234532345234

Chapter 9: Non-versatility of MAX

It is the purpose of the present chapter to show that none of the digraphs of MAX are versatile. We commence by proving a useful theorem. First we make a definition:

Definition: Suppose that v is a non-repetitive word of type ω, on alphabet $\Sigma = \{ a_1, a_2, \ldots, a_n \}$. An a_1-block is a subword w of v so that a_1 is a prefix of w, w contains exactly one a_1, and w appears in the context wa_1 in v.

Block Theorem: Let a, b, c, d be words over some alphabet Σ such that b is a prefix of c, which is a prefix of d. Then a, b, c, d cannot be concatenated to form a non-repetitive word of type ω.

Proof: Suppose that a, b, c, d could be concatenated to form a non-repetitive word v of type ω. Suppose that the word a does not appear in v infinitely often; then word b never occurs, as bc and cd contain repetitions. But then c never occurs, for c cannot be followed by d in a non-repetitive word. This leaves the single word d, which of course cannot be concatenated with itself to form any non-repetitive words.

Thus we may assume that v contains the word a. Since v is an ω word, assume without loss of generality that v commences with the word a. In fact assume without loss of generality that every one of the words a, b, c, d appearing in v occurs infinitely often in v. We may think of v as a "meta-word", whose letters are a, b, c, d. If we parse v, chopping it into pieces at each occurrence of a, the possible a-blocks are:

ab

ac

B: ad

A: acb

D: adc

C: adb

E: adcb

We never, of course, find subwords bc, bd, cd in v, as these contain repetitions. Moreover, of these a-blocks, only A, B, C, D, and E can appear infinitely often in v; the piece ab is a prefix of the other pieces, and thus never appears in v. (What would follow it in v?) Again, once we have disposed of piece ab, piece ac is a prefix of all the other pieces and cannot be used either. Thus v is concatenated from pieces A, B, C, D, E. We assume without loss of generality that each of these pieces appearing in v does so infinitely often.

The eccentric lettering of these pieces (B, A, D, C, E) simply makes note of the fact that B is a prefix of C, which is a prefix of D, a prefix of E. We now take our argument one level deeper; as B is a prefix of C, which is a prefix of D, a prefix of E, v must contain block A. Parse v by chopping it up wherever the piece A appears followed by an a. Offhand, we get several pieces. However some of these A-blocks can only appear finitely often in v, and can hence without loss of generality, be assumed not to occur in v.

<u>AB</u> (1), <u>AC</u> (1), <u>AD</u> (4), AE

<u>AED</u> (2), <u>AEC</u> (2), <u>AEB</u> (2), ADC, ADB, <u>ACB</u> (3)

<u>AEDC</u> (2), <u>AEDB</u> (2), <u>AECB</u> (2), ADCB

<u>AEDCB</u> (2)

<u>Notes</u>:(1) As the block AB is a prefix of all the other blocks, it cannot appear in v. However, the block AC is a prefix of every block but AB, and hence AC cannot appear in v either.

(2) Here AEX (where X is B, C or D) will contain cb adcb ad, a repetition. Thus no block containing such a

word can appear in v.

(3) Here ACB contains the repetition badbad.

(4) After the eliminations of (1) and (3), AD is a prefix of the remaining blocks, and must be discarded.

We are left with four A-blocks to concatenate to form v:

α: AE

β: ADB

γ: ADC

δ: ADCB

We have almost come full circle; here β is a prefix of γ, a prefix of δ. Again α must appear in v. However, we now have quite a lot of conditions on α, β, γ, δ. By our examination of a-blocks, we know that the blocks resulting when v is chopped into pieces at α are:

$\alpha\delta$

$\underline{\alpha\gamma\beta}$

$\alpha\delta\gamma$

$\alpha\delta\beta$

$\alpha\delta\gamma\beta$

However here $\alpha\gamma\beta \supset$ E ADC ADB $\supset$ bADadbADad, a repetition, so that the block $\alpha\gamma\beta$ can never appear in v. However, once $\alpha\gamma\beta$ has been discarded, $\alpha\delta$ is a prefix of the four remaining blocks and must also be discarded. This leaves three blocks, $\alpha\delta\beta$, $\alpha\delta\gamma$, $\alpha\delta\gamma\beta$, with the first a prefix of the other two. One checks that there are no non-repetitive words of length greater than three on two letters. Since v therefore could not be formed from two blocks, we have a contradiction. Thus words a, b, c, d cannot be concatenated to form a non-repetitive word of type ω.□

Using this Block Theorem, and similar arguments, we show that none of the digraphs of MAX is versatile.

<u>MAX.1</u>

Suppose that we could walk some non-repetitive word v of type ω on MAX.1. If v contains no 2, then v can be walked on one of the strongly connected components of MAX.1 \ { 2 }. However none of these components has more than two vertices, so that this is impossible.

Parse v by chopping it into pieces commencing with 2. The possible 2-blocks on MAX.1 are:

a: 21

b: 23

c: 2345

d: 234565

By the Block Theorem, these words cannot be concatenated to form v. This is a contradiction and we conclude that MAX.1 is not versatile.

MAX.2

The proof that MAX.2 is not versatile is more involved. Suppose that MAX.2 is versatile, and let v be a non-repetitive word of type ω walkable on MAX.2. If v contains no 3, then v is walked on one of the strongly connected components of MAX.2 \ { 3 }, which is impossible, as each of these components consists of a single vertex. By analyzing the 3-blocks of v which could be walked on MAX.2, we can conclude that MAX.2 is not versatile.

Level 1: 3-blocks:

a: 34567

b: 345612

c: 3452

d: 342

e: 32

One checks that these are all the 3-blocks on MAX.2.

Next, looking at v as composed of letters a, b, c, d, e, we look at b-blocks. (Note that v must contain a b; otherwise v is composed of blocks a, c, d, e. This possibility is not excluded by the Block Theorem above, however the proof that it cannot occur follows almost exactly the proof of the Block Theorem, and is therefore here omitted.) Here are the b-blocks of which v could be composed:

Level 2: b-blocks:

ba, bc (1), bd (1), be (1)

The underlined two-letter words never appear in v, as will be shown in note (1) of the comments below. We therefore omit looking at any blocks on three or more letters that contain these words. Call a word which cannot appear in v more than finitely often **illegal**. Clearly no block containing an illegal word can appear in v more than finitely often, so that such blocks may be discarded without loss.

bac, bad, bae

baca, **bacd** (1), **bace** (1), bada, **badc** (1), bade (1), baea (2), baec (1), baed (1)

Again, the underlined blocks are illegal, and this decreases the number of blocks on five or more letters we need to examine.

bacac (3), bacad, bacae, badac (4), badad (5), badae

bacada, bacadc (1), bacade (1), bacaea (2), bacaec (1), bacaed (1), badaea (2), badaec (1), badaed (1)

bacadac (4), bacadad (5), bacadae

bacadaea (2), bacadaec (1), bacadaed (1)

Notes on the b-blocks

(1) The words 2e3, 2d34, 2c345 are repetitions, hence illegal. Since each b-block starts with 3, we therefore see that b-blocks containing the piece 2e may be discarded. Thus the words de, ce , be are illegal. This means in particular that d, whenever it appears in v, is

always followed by a b-block commencing 34.

It follows that ed, cd, bd are illegal.

As cd and ce are illegal, c is always followed by 345 in any b-block. Thus ec, dc, bc are illegal.

(2) The particle aea is always preceded by a 2 and followed by a three, and thus contains 2a3 2a3. Thus aea is illegal.

(3) Here acac is a repetition. We thus discard any block containing acac.

(4) The block contains 2a34 2a34.

(5) Here adad is a repetition.

Summary of usable b-blocks

Let us here list the b-blocks which we have not yet eliminated:

ba (1), bac, bad (6), bae (5), baca (3), bada (2), bacad, bacae, badae, bacada (4), bacadae

Again, many of these blocks must be discarded.

(1) The block ba is a prefix of all the other useful b-blocks, and thus can't be followed by any of them in a

non-repetitive word.

(2) Leads to bada 34 $\supset$ 2a34 2a34.

(3) Gives baca 345 $\subset$ 2a345 2a345

(4) Here this block appears only in the context bacada 34 which contains 2ada34 = 2a34 2a34.

(5) After the elimination of ba, baca, bada, bacada, the only context in which this block could appear is 2 bae ba3 = 2ba3 2ba3.

(6) Now this block appears only in context 2badba34 = 2ba34 2ba34.

We are left with five b-blocks:

A: bac

B: badae

C: bacad

D: bacae

+: bacadae

After one more level of blocks, we are done. Note that $A < C < D < E$ in the sense that the words AC, AD, AE, CD, CE, DE are illegal, so that v must contain a B.

<u>Level 3: B-blocks:</u>

BA, BC, BD, BE,

BAC (1), BAD (1), BAE (1), BCA, BCD, BCE (2), BDA (3), BDC (3),

BDE (3), BEA (4), BEC (4), BED (4)

BCAC (1), BCAD (1), BCAE (1), BCDA, BCDC (5), BCDE (5)

BCDAC (1), BCDAD (1), BCDAE (1)

Notes on the B-blocks

(1) A is a prefix of C, D, E.

(2) C is a prefix of E.

(3) Here BDbac ⊃ adaebac adaebac, so that BDX is illegal, where X is A, C or E.

(4) BEbac ⊃ adaebac adaebac.

(5) Block contains CDbacad ⊃ 2baca3 2baca3.

Summary of useful B-blocks

BA (1)

BC (4)

BD (2)

BE (3)

α: BCA

β: BCD

γ: BCDA

Notes: (1) Must be discarded, since it is a prefix of the others.

(2) Once (1) is gone, this block always appears in the context 2BDBbaca ⊃ 2Bbaca3 2Bbaca3.

(3) This word EBb is illegal, as it contains adaeb adaeb.

(4) After (1), (2), (3) are gone, this block is a prefix of the remaining blocks.

We are thus left with blocks α, β, γ with which to form a non-repetitive word of type ω. However, as β is a prefix of γ, $\beta\gamma$ contains a repetition. Recall from our remarks in Chapter 1 that if α, β, γ can be concatenated to form a non-repetitive ω word, then $\alpha\beta$, $\alpha\gamma$, $\beta\alpha$, $\beta\gamma$, $\gamma\alpha$, $\gamma\beta$ must each be non-repetitive, We thus conclude that MAX.2 is not versatile.

Having given some details for MAX.1, MAX.2, we give less detail for the other cases, as there are, after all, 26 digraphs in MAX.

MAX.3

Suppose that v is a non-repetitive ω word on MAX.3 Then v must contain a 3, since MAX.3 \ { 3 } has only trivial strongly connected components.

Level 1: 3-blocks:

a: 32

b: 342

c: 34512

d: 3456

e: 34562

Suppose that v contains no d. Then a occurs only as 2a3, a repetition. Thus v contains no a. With a, d excluded, b must occur as 2b34 , a repetition, so that v contains no d, b or a. This is impossible.

Level 2: d-blocks:

da, db, dc, de (2)

dab (1), dac, dae, dba (1). dbc, dbe, dca (1), dcb (1), dce

daca (1), dacb (1), dace, daea (1), daeb (1), daec, dbca (1), dbcb (1), dbce, dbea (1), dbeb (1), dbec, dcea (1), dceb (1), dcec

dacea (1), daceb (1), dacec, daeca (1), daecb (1), daece, dbcea (1), dbceb (1), dbcec, dbeca (1), dbecb (1), dbece, dceca (1), dcecb (1), dcece (3)

daceca (1), dacecb (1), dacece (3), daecea (1), daeceb (1), daecec (4), dbceca (1), dbcecb (1), dbcece (3), dbecea (1), dbeceb (1), dbecec (4)

Notes on the d-blocks

(1) The word 2a3 is a repetition. Since each d-block starts with 3, we therefore see that the combinations ba, ca, da cannot appear. This means that b, whenever it appears in a non-repetitive of type ω, is always followed by a block commencing 34. However, 2 b 34 is a repetition. Thus blocks ab, cb, eb must not be used.

(2) The block d is a prefix of e.

(3) Here ce repeats.

(4) The block contains ec ec.

Summary of useful d-blocks

da (1), db, dc (4),

dac, dae (2), dbc, dbe (2), dce (2)

dace (2), daec, dbce (2), dbec, dcec (5)

dacec (3), daece (2), dbcec (3), dbece (2),

Notes

(1) Leads to 2 da d3 = 2d32d3, a repetition.

(2) As the word 2e d is a repetition.

(3) Since 2cec d ⊃ 2cd 2cd, a repeat.

(4) After eliminations (1) - (3), all remaining blocks except for db end in c. Thus dc occurs in context 2 db dc ⊃ 2d342d34, or in context c dc d, both repetitions.

(5) As in elimination (4), this block is preceded either by db or c, giving rise to word 2 db dc, which contains a repetition, or c dce ⊃ cdcd.

We are left with five d-blocks:

A: db

B: dbc

C: dbec

D: dac

E: daec

After one more level of blocks, we are done. If D does not occur in v, then v is concatenated from A, B, C, E, and A < B < C in the sense that AB, AC, BC are illegal. Here BC is illegal because it must occur in the context cBC ⊃ cdbcdb. Arguing analogously to the proof of the Block Theorem, we get a contradiction.

<u>Level 3: D-blocks:</u>

DA, DB, DC, DE

DAB (1), DAC (1), DAE, DBA (1), DBC (1), DBE, DCA, DCB, DCE (1), DEA, DEB, DEC (1)

DAEA, DAEB, DAEC (1), DBEA, DBEB, DBEC (1), DCAB (1), DCAC (1), DCAE, DCBA (1), DCBC (1), DCBE, DEAB (1), DEAC (1), DEAE, DEBA (1), DEBC (1), DEBE

DAEAB (1), DAEAC (1), DAEAE (2), DAEBA (1), DAEBC (1), DAEBE, DBEAB (1), DBEAC (1), DBEAE, DBEBA (1), DBEBC (1), DBEBE (3), DCAEA, DCAEB, DCAEC (1), DCBEA, DCBEB, DCBEC (1), DEAEA (4), DEAEB (4), DEAEC (1), DEBEA (5), DEBEB (6), DEBEC (1)

DAEBEA, DAEBEB (6), DAEBEC (1), DBEAEA (4), DBEAEB (4), DBEAEC (1), DCAEAB (1), DCAEAC (1), DCAEAE (2), DCAEBA (1), DCAEBC (1), DCAEBE, DCBEAB (1), DCBEAC (1), DCBEAE, DCBEBA (1), DCBEBC (1), DCBEBE (3)

DAEBEAB (1), DAEBEAC (1), DAEBEAE, DCAEBEA, DCAEBEB (6), DCAEBEC (1), DCBEAEA (4), DCBEAEB (4), DCBEAEC (1)

DAEBEAEA (4), DAEBEAEB (4), DAEBEAEC (1), DCAEBEAB (1), DCAEBEAC (1), DCAEBEAE

DCAEBEAEA (4), DCAEBEAEB (4), DCAEBEAEC (1)

Notes on the D-blocks

(1) A is a prefix of B, C. B is a prefix of C. The word cBA is a repetition, and also appears in cBC. The word CE is illegal as it appears in the context CEd3 $\supset$ 2ecd32ecd3. Also CD is illegal, leading to CDd3 $\supset$ 2cd32cd3. Thus CE is always followed by A or B, and hence db. The word EC therefore appears in context ECdb $\supset$ ecdbecdb.

(2) Contains AEAE.

(3) Contains BEBE.

(4) Contains EAEA.

(5) Contains cEBEA = c E db c E db.

(6) Contains EBEB.

Summary of useful D-blocks

DA, DB (1), DC (1), DE (1)

DAE (1), DBE (1), DCA, DCB (1), DEA, DEB (1)

DAEA, DAEB (1), DBEA, DBEB (1), DCAE (1), DCBE (1), DEAE (1), DEBE (1)

DAEBE (1), DBEAE (1), DCAEA, DCAEB (1), DCBEA, DCBEB (1)

DAEBEA, DCAEBE (1), DCBEAE (1),

DAEBEAE (1), DCAEBEA,

DCAEBEAE (1)

Notes: (1) A combination (d-block other than A) D d3 will contain 2c d32c d3, a repetition.

Summary of remaining blocks

DA (1)

DCA (3)

DEA (2)

DAEA (4)

DBEA (6)

DCAEA (5)

α: DCBEA

β: DAEBEA

γ: DCAEBEA

Notes

(1) Appears as A DA D.

(2) After (1) is gone, appears as 2ecA DEA Dd3, a repetition.

(3) After (2) is gone, this block always appears in the context

EA DCA Ddb $\supset$ ec A D dbec A D db.

(4) After eliminations (1) - (3), this block appears only as EA DAEA Ddb, a repetition.

(5) Here DCAEA D3 $\supset$ 2ec A d32ec A d3, a repeat.

(6) After (1) - (5) are eliminated, this block appears

only as

BEA DBEA D.

We are thus left with blocks α, β, γ with which to form a non-repetitive word of type ω. However, it follows from our remarks in Chapter 1, in the first open problem, that if v is a non-repetitive word concatenated from α, β, γ, then v must contain all of the three letter subwords $\alpha\beta\gamma$, $\alpha\gamma\beta$, $\beta\alpha\gamma$, $\beta\gamma\alpha$, $\gamma\beta\alpha$, $\gamma\alpha\beta$. We conclude that MAX.3 is not versatile.

MAX.4

Suppose that MAX.4 is versatile, and let v be a non-repetitive word of type ω walkable on MAX.4. One checks that MAX.4 \ {1} may be reduced to the three element path, and is not versatile. Therefore v must contain a 4. By analyzing the 1-blocks of MAX.4, we can conclude that MAX.4 is not versatile.

Level 1: 1-blocks:

12345 (1)

1234565 (2)

a: 1234562345

b: 123456345

12345632345 (3)

c: 1234563234565

d: 1234563234562345

Notes on the 1-blocks

(1) This block is a prefix of the others, and hence must be discarded.

(2) After (1) is gone, this block always appears in the context 5 1234565 123456, which is a repetition.

(3) This block is a prefix of c and d, and hence cannot be followed by them in a non-repetitive word. However, 2345 a 123456 is a repetition, as is 345 b 123456, so that the block must be discarded once (1) and (2) are gone.

We must include block c in v; otherwise block b can only occur in the context 345 b 123456, which is impossible. We are then left with only the two blocks a and d.

Level 2: c-blocks:

ca, cb, cd (1)

cab (2), cad (3), cba, cbd

cbab (2), cbad (3), cbda (3), cbdb (4)

Notes on the c-blocks

(1) The word 5cd is a repetition.

(2) Since ab123456 contains a repetition.

(3) The word da is illegal. Thus ad is illegal as ada is illegal and adb, adc contain repetitions.

(4) The word db is illegal as it appears in the context db123456 ⊃ 345b123456, which is a repetition.

Summary of useful c-blocks

α: ca

cb (1)

β: cba

γ: cbd

(1) 345 cb c123456 = 345 c 123456 345 c 123456

The three blocks α, β, γ cannot be concatenated to form a non-repetitive word of type ω, since $\alpha\beta\gamma$ contains a repetition. Thus MAX.4 is not versatile.

MAX.5

Suppose that MAX.5 is versatile, and let v be a non-repetitive word of type ω walkable on MAX.5. One checks as in the previous case that 1 occurs in v.

Level 1: 1-blocks:

123456 (1)

12345676 (2)

a: 123456756

b: 12345673456

1234567323456 (3)

c: 123456732345676

d: 1234567323456756

Notes on the 1-blocks

(1) This block is a prefix of the others, and is discarded.

(2) After (1) is gone this block always appears in context 6 12345676 1234567, a repetition.

(3) This block is a prefix of c and d, and hence cannot be followed by them in a non-repetitive word. However, 56 a 1234567 is a repetition, as is 3456 b 1234567, so that the block is discarded.

The word bac contains a repetition, so that we cannot concatenate v from a, b, c alone.

Level 2: d-blocks:

da (1), db, dc (2)

dba (1), dbc

dbca, dbcb

dbcab, dbcac (4), dbcba (1), dbcbc (3)

dbcaba (1), dbcabc (5)

Notes on the d-blocks

(1) The word 56 a 1234567is a repetition so that ba and da are illegal..

(2) Since 56dc is a repetition.

(3) Here bcbc is a repetition.

(4) Since dbcac 1234567 ⊃ 56c1234567 56c1234567, a repetition.

(5) Here dbcabc 1234567 will contain the repetition 56bc1234567 56bc1234567.

Summary of useful d-blocks

db (1)

dbc (2)

α: dbca

β: dbcb

γ: dbcab

(1) A prefix of the remaining blocks.

(2) A prefix to the blocks remaining after the elimination of (1).

The three blocks α, β, γ cannot be concatenated to form a non-repetitive word of type ω, since αγ contains a repetition. Thus MAX.5 is not versatile.

MAX.6

Suppose that MAX.6 is versatile, and let v be a non-repetitive word of type ω walkable on MAX.6. As in the previous two cases, v must contain a 1.

Level 1: 1-blocks:

123 (1)

1234 (2)

a: 123454

b: 123453

c: 1234534

d: 12345323

e: 123453234

f: 12345323454 (3)

Notes on the 1-blocks

(1) This block is a prefix of the others.

(2) After (1) is gone, this block is a prefix of the others.

(3) The words bf, cf, df, ef, af are illegal. This last, af, is illegal since fa is illegal and af123453 ⊃ 2345412345312345412345З.

Suppose that c does not occur in v. Then v is formed from a, b, d, e, and b is a prefix of d, a prefix of e, a prefix of f. This is impossible by the Block Theorem.

To show that MAX.6 is not versatile, we consider blocks ending in c. Note that since cf contains a repetition

Level 2: reverse c-blocks:

ac (2), bc (1), dc, ec (2),

adc, bdc (1), edc (3),

badc, dadc (4), eadc (5),

abadc (6), dbadc (7), ebadc,

aebadc, bebadc (1), debadc (1)

baebadc (8), daebadc, eaebadc (5)

adaebadc (9), bdaebadc (1), edaebadc (3)

Notes on the reverse c-blocks

(1) The words bc, bd, be, bf, de, df, ef contain repetitions.

(2) Here 4c gives either 4 c 123453, a repetition, or 4 c a 12345 ⊃ 4 a 12345, a repetition.

(3) Here ed leads to one of be, de, or 4 ed, each repetitions.

(4) Leads to bd or 4dad12345.

(5) As 4 a 12345 is a repetition.

(6) Since ab is repeated.

(7) Contains 312345312345.

(8) We get 3b12345 or 4 bae 4 bad.

(9) Since ad is repeated.

Summary of useful reverse c-blocks

dc (1)

adc (2)

badc (3)

ebadc (4)

aebadc (2)

daebadc

(1) A suffix of the remaining blocks.

(2) As ca leads to 4 123454 12345.

(3) After (1) - (2) are gone, this block is a suffix of the remaining blocks.

(4) Appears in the context 4badc ebadc d, a repetition.

MAX.6 is not versatile.

MAX.7.

Suppose that MAX.7 is versatile, and let v be a non-repetitive word of type ω walkable on MAX.7. Again, MAX.7 \ {1} reduces to a three element path. The 1-blocks of MAX.7 are

a: 1232

b: 123456

c: 12345676

d: 1234567656

Thus by the Block Theorem, MAX.7 is not versatile.

MAX.8

Suppose that MAX.8 is versatile, and let v be a non-repetitive word of type ω walkable on MAX.8. We check that MAX.8 \ {4} reduces to a three element path. The 4-blocks of MAX.8 are

a: 4123

b: 45

c: 4563

d: 456323

Thus by the Block Theorem, MAX.8 is not versatile.

MAX.9

Suppose that MAX.9 is versatile, and let v be a non-repetitive word of type ω walkable on MAX.9. We check that MAX.9 \ {1} is mimicked by MAX.7, which has been shown not to be versatile. One may therefore assume that v contains a 1.

Level 1: 1-blocks:

12345 (1)

a: 1234562345

12345645 (2)

b: 1234564345

c: 12345643456 2345 (3)

d: 1234564345623456 45

e: 12345643456234564345

Notes on the 1-blocks

(1) This block is a prefix of the others.

(2) After (1) is gone, this block always appears in the context 45 12345645 123456, which is a repetition.

Level 2: a-blocks:

ab, ac, ad, ae

abc (1), abd (1), abe (1), acb (2), acd (1), ace (1), adb, adc, ade (1), aeb (3), aec, aed (4)

adbc (1), adbd (1), adbe (1), adcb, adcd (1), adce (1), aecb, aecd (1), aece (1) adcbc (1), adcbd (1), adcbe (1), aecbc (1),

aecbd (1),

aecbe (1)

Notes on the a-blocks

(1) The words bc, bd, be, cd, ce, de are illegal.

(2) Since acb ⊃ 62345b 62345b.

(3) The word aebl contains 2345643451 2345643451.

(4) The word aed contains a repetition of 34512345643456234564.

Summary of useful a-blocks

ab (2)

ac (1)

ad (3)

ae (4)

adc (1)

α: adb

aec (1)

β: adcb

γ: aecb

(1) ca 123456 $\supset$ 2345 1234562345 123456.

(2) Prefix of other blocks.

(3) After elimination (2), the a-block ad must occur in the context 5adae, which contains a repetition..

(4) Among the remaining blocks, ae appears either in the context aeae, or as b ae ad $\supset$ (345 a 12345643456234564)^2.

The three blocks α, β, τ cannot be concatenated to form a non-repetitive word of type ω, since $\alpha\beta$ contains a repetition. Thus MAX.9 is not versatile

MAX.10

Suppose that MAX.10 is versatile, and let v be a non-repetitive word of type ω walkable on MAX.10. One checks that vertex 4 must occur in v. However, MAX.10 is versatile if and ony if its reverse is. The 4-blocks of the reverse of MAX.10 are

a: 43

b: 432

c: 43215

d: 465

Invoking the Block Theorem, MAX.10 is not versatile.

MAX.11

One checks that 4 cannot be discarded from MAX.11. The 4-blocks of the reverse of MAX.11 are

a: 432

b: 4321

c: 43212

d: 45

This is impossible by the Block Theorem. Thus MAX.11 is not versatile.

MAX.12

Suppose that MAX.12 is versatile, and let v be a non-repetitive word of type ω walkable on MAX.12. One checks that 1 must appear in v. The 1-blocks of MAX.12 are

12

123

12345

a: 1234565

b: 1234562

c: 12345623

d: 1234562345

The block 12 is not useful, as it is a prefix of the other blocks. Similarly, the blocks 123, 12345 are eliminated from consideration. However, by the Block Theorem, the remaining blocks cannot be concatenated to form a non-repetitive word of type ω. Thus MAX.12 is not versatile.

MAX.13

Suppose MAX.13 is versatile for some q. Let v be a non-repetitive word of type ω walkable on MAX.13. One checks that we may assume that v contains a 1. The 1-blocks of MAX.13 are:

12

123

1234

...

1234...(q-1)

123...q2

123...q23

123...q234

...

123...q234...(q-1)

We see that the block 12 is a prefix of all the other blocks, and hence cannot appear in v. Again, the block 123 is a prefix of all the other blocks excluding 12, and hence can never be used in v. Continuing in this way, we can eliminate all the blocks in order, showing that none of them can be used in v, which is a contradiction. Thus MAX.13 is not versatile for any q.

MAX.14

Suppose that MAX.14 is versatile, and let v be a non-repetitive word of type ω walkable on MAX.14. One checks that v may be assumed to commence with 1. The 1-blocks of MAX.14 are

12, which is discarded

a: 1232

b: 1234

c: 123456

d: 12345676

By the Block Theorem, these blocks cannot be concatenated to form a non-repetitive word of type ω. Thus MAX.14 is not versatile.

MAX.15

Suppose that MAX.15 is versatile, and let v be a non-repetitive word of type ω walkable on MAX.15. One checks that v may be assumed to contain a 5. The 5-blocks of MAX.15 are

a: 56

b: 51234

c: 512343234

Since bc contains a repetition, these blocks cannot be concatenated to form a non-repetitive word of type ω. Thus MAX.15 is not versatile.

MAX.16

Suppose that MAX.16 is versatile, and let v be a non-repetitive word of type ω walkable on MAX.16. One checks that v may be assumed to contain a 3 .The 3-blocks of MAX.16 are

a: 312

b: 342

c: 345

Since abc contains a repetition, these blocks cannot be concatenated to form a non-repetitive word of type ω. Thus MAX.16 is not versatile.

MAX.17

Suppose that MAX.17 is versatile, and let v be a non-repetitive word of type ω walkable on MAX.17. One checks that v may be assumed to contain a 3.

Level 1: 3-blocks:

a: 32

b: 312

c: 3412

d: 345

e: 34512

One checks that v may be assumed to contain a d.

Level 2: d-blocks:

da, db, dc, de (1)

dab, dac, dae. dba (2), dbc (3), dbe (5), dca

(2), dcb (4), dce (5)

daba (2), dabc (3), dabe (5), daca (2), dacb (4),
dace (5), daea (2), daeb (4), daec (3)

<u>Notes on the d-blocks</u>

(1) The word de contains a repetition.

(2) Since 2a3 is a repetition.

(3) Here 12c will either appear in context

12c34, a repetition,

12cb3 ⊃ 12b3, a repetition,

or 12ca3 ⊃ 2a3, a repetition.

(4) As noted above, 12b3 is a repetition.

(5) Here 12e will appear in one of the following contexts:

12 e 345 , a repetition,

12 e c ⊃ 12 c, leading to a repetition as in (3) above,

12 e a ⊃ 2 a, leading to repetition as in (2)

or 12 e b ⊃ 12 b, leading to the repetition of (4).

<u>Summary of useful d-blocks</u>

da (1)

db (2)

A: dc

B: dab

C: dac

D: dae

(1) 2dad3 is a repetition.

(2) After the elimination (1), this block appears in context

12 db d3, a repetition

Thus if MAX.17 is versatile, a non-repetitive word can be composed of blocks A,B,C,D. However this would imply that the following blocks could be concatenated to form a non-repetitive word of type ω:

A': 12d34

B': 12da3

C': 12da34

D': 12da345

As this is impossible by the Block Theorem, MAX.17 is not versatile.

MAX.18

Suppose that MAX.18 is versatile, and let v be a non-repetitive word of type ω walkable on MAX.18. One checks that v may be assumed to contain a 1.

Level 1: 1-blocks:

12345 (1)

123456 (2)

a: 1234565

b: 123456345

12345632345 (3)

c: 123456323456

d: 1234563234565

Notes on the 1-blocks

(1) This block is a prefix of the others.

(2) After (1) is gone, this block is a prefix of the remaining blocks.

(3) This block is a prefix of c and d, and hence cannot be followed by them in a non-repetitive word. However, 5 a 123456 is a repetition, as is 345 b 123456, so that this block is also is discarded.

One checks that v may be assumed to contain a c.

Level 2: c-blocks:

ca, cb, cd (1)

cab, cad (2), cba (1), cbd

caba (1), cabd, cbda (1), cbdb

cabda (1), cabdb, cbdba (1), cbdbd (3)

cabdba (1), cabdbd (3)

Notes on the c-blocks

(1) The word 5a leads to a repetition 5 1234565 123456, so that words ba, da are not useful. The word cd contains a repetition.

(2) Since da is illegal, so is ad, which must occur in the context adx, x = b or c, hence ad1234563 ⊃ 2345651234563234565123456З.

(3) Repeats bd.

Summary of useful c-blocks

ca (2)

cb (3)

cab (4)

cbd (1)

cabd (1)

cbdb

cabdb

(1) As 5dc ⊃ 5 1234563234565 123456323456.

(2) For, 5 cac 123456 ⊃ 5c1234565c123456.

(3) After (1), (2) are gone, cb always appears in context b cb c.

(4) Here db cab c 1 ⊃ 234565 b c 1234565 b c 1.

We are left with only two blocks. Thus MAX.18 is not versatile.

<u>MAX.19</u>

Suppose that MAX.19 is versatile, and let v be a non-repetitive word of type ω walkable on MAX.19. One checks that v may be assumed to contain a 1.

Level 1: 1-blocks:

1234 (1)

12345 (2)

a: 12345234

b: 1234534

123453234 (3)

c: 1234532345

d: 1234532345234

Notes on the 1-blocks

(1) This block is a prefix of the others.

(2) After (1) is gone, this block is a prefix of the remainder.

(3) This block is a prefix of c and d, and hence cannot be followed by them in a non-repetitive word. However, 234 a 12345 is a repetition, as is 34 b 12345, so that the block is discarded.

One checks that v may be assumed to contain a c.

Level 2: c-blocks:

ca, cb, cd (1)

cab (2), cad (3), cba (1), cbd

cbab (2), cbad (3), cbda (4), cbdb (2)

Notes on the c-blocks

(1) The word cd contains a repetition.

(2) Since 34 b 12345 is a repetition.

(3) Here ad leads to ad123453, a repetition, or adb12345, a repetition. (See (2).)

(4) Since 234 a 12345 is a repetition.

Summary of useful c-blocks

cb (1)

α: ca

β: cba

γ: cbd

(1) As 34 cb c 12345 is a repetition.

The three blocks α, β, γ cannot be concatenated to form a non-repetitive word of type ω, since $\beta\alpha\gamma$ contains a repetition. Thus MAX.19 is not versatile.

MAX.20

Suppose that MAX.20 is versatile, and let v be a non-repetitive word of type ω walkable on MAX.20. One checks that v may be assumed to contain a 6, since MAX.20 \ {6} is mimicked by MAX.19.

Level 1: 6-blocks

612345 (1)

61234532345 (2)

a: 612345323412345

b: 6123453412345

c: 612345341234532345

d: 6123453412345323412345

e: 61234534123453234123453412345

Notes on 6-blocks

(1) This block is a prefix of the others.

(2) After elimination (1), this block always occurs in the context 23456 12345323456 1234563, which is a repetition.

One checks that v may be assumed to contain an a.

Level 2: a-blocks:

ab (1), ac, ad (2), ae

acb, acd (2), ace (2), aed (2), aec, aeb (2)

aecb, aecd (2), aece (2), aebc (2), aebd (2), aebe (2)

aecbc (2), aecbd (2), aecbe (2)

Notes on the a-blocks

(1) The word ab always occurs in the context ab 612345 $\supset$ 3412345 612345 3412345 612345, a repetition.

(2) The following words contain repetitions:

da6, ada6, ad61234534

(Thus ad is illegal.)

bc, bd, be, cd, ce, de

eb6, aed.

Summary of useful a-blocks

ac (1)

B: ae

A: acb

C: aec

D: aecb

(1) As 5acae is illegal.

By the Block Theorem, MAX.20 is not versatile.

MAX.21

Suppose that MAX.21 is versatile, and let v be a non-repetitive word of type ω walkable on MAX.21. One checks that v may be assumed to contain a 1. The 1-blocks of MAX.21 are

123 (1)

1234 (2)

a: 12345

b: 12343

c: 1234323

d: 12343234

123432345 (3)

Notes: Blocks (1) and (2) are eliminated as prefixes. Block (3) cannot be preceded by any block but a, and thus must appear in one of two contexts:

a 123432345 12343 ⊃ 234512343 234512343

a 123432345 12345 1 ⊃ 23451 23451.

By the Block Theorem, the remaining blocks cannot be concatenated to form a non-repetitive word of type ω. Thus MAX.21 is not versatile.

<u>MAX.22</u>

Suppose that MAX.22 is versatile, and let v be a non-repetitive word of type ω walkable on MAX.22. One checks that v may be assumed to contain a 1. The 1-blocks of MAX.22 are

12

123

1234

a: 123456

b: 123452

c: 1234523

~~d:~~ 12345234

The first three blocks are eliminated in turn, as prefixes. By the Block Theorem, the remaining blocks cannot be concatenated to form a non-repetitive word of type ω. Thus MAX.22 is not versatile.

MAX.23

Suppose that MAX.23 is versatile, and let v be a non-repetitive word of type ω walkable on MAX.23. One checks that v may be assumed to contain a 1. The 1-blocks of MAX.23 are

12, which is discarded

a: 1232

b: 12345

c: 123456

d: 1234565

By the Block Theorem, these blocks cannot be concatenated to form a non-repetitive word of type ω. Thus MAX.23 is not versatile.

MAX.24

Suppose that MAX.24 is versatile, and let v be a non-repetitive word of type ω walkable on MAX.24. One checks that v may be assumed to contain a 1. The 1-blocks of MAX.24 are

12, which is discarded

a: 1232

b: 1234

c: 12345

d: 123456

By the Block Theorem, these blocks cannot be concatenated to form a non-repetitive word of type ω. Thus MAX.24 is not versatile.

MAX.25

Suppose that MAX.25 is versatile, and let v be a non-repetitive word of type ω walkable on MAX.25. One checks that v may be assumed to contain a 2. By analyzing the 2-blocks of MAX.25, we can conclude that MAX.25 is not versatile.

Level 1: 2-blocks:

a: 21

b: 234

c: 23454

d: 23451

One checks that v may be assumed to contain an a.

Level 2: a-blocks:

ab, ac, ad

abc (1), abd (1), acb, acd, adb, adc (2)

acbc (1), acbd (2), acdb, acdc, adbc (1), adbd (2)

acdbc (1), acdbd (1), acdcb, acdcd (3)

acdcbc (1), acdcbd (1)

Notes on the a-blocks

(1) The word b is a prefix of c and d.(2) This block followed by 3 contains 2a3 2a3.

(2) Here adc ⊃ 1234512345.

(3) The word cd repeats.

Summary of useful a-blocks

ab (2)

F: ac

ad (1)

A: adb

      acd (1)

B: acb

C: acdb

D: acdc

E: acdcb

Notes

(1) The block da2 ⊃ 1212.

(2) A prefix of the other blocks.

We are left with five a-blocks:

After one more level of blocks, we are done.

Level 3: A-blocks:

AB, AC, AD, AE, AF (1)

ABC (2), ABD (2), ABE (2), ACB, ACD (3), ACE (3), ADB, ADC (6),

ADE (4), AEB (5), AEC, AED (7)

ACBC (2), ACBD (2), ACBE (2), ADBC (2), ADBD (2), ADBE (2), AECB,

AECD (3), AECE (3),

AECBC (2), AECBD (2), AECBE (2).

Notes on the A-blocks

(1) A prefix of the other blocks.

(2) B is a prefix of C, D, E.

(3) C is a prefix of D, E.

(4) D prefix of E.

(5) Here AEBa ⊃ cb acb a.

(6) As DC contains dcac2 ⊃ 1c21c2.

(7) As AED ⊃ b acdcb acdc.

Summary of useful A-blocks

AB (1)

AC (2)

AD (6)

AE

ACB (5)

ADB (4)

AEC (3)

AECB

Notes: (1) Prefix.

(2) Once (1) is gone, this block is a prefix.

(3) Leads to CA a ⊃ db adb a

(4) DB ⊃ dcac2 ⊃ 1c2 1c2.

(5) ACB ⊃ db acdb ac.

(6) After (1) - (5) are gone, this block is prefix of the remainder.

We are thus left with only two blocks. These blocks cannot be concatenated to form a non-repetitive word of type ω.

<u>MAX.26</u>

Suppose that MAX.26 is versatile, and let v be a non-repetitive word of type ω walkable on MAX.26. One checks that v may be assumed to contain a 6. The 6-blocks of MAX.26 are

67, which is discarded

a: 6787

b: 67812345

c: 678123454345

d: 6781234543452345

By the Block Theorem, these blocks cannot be concatenated to form a non-repetitive word of type ω. Thus MAX.26 is not versatile.

We have now established that none of the digraphs of MAX are versatile.

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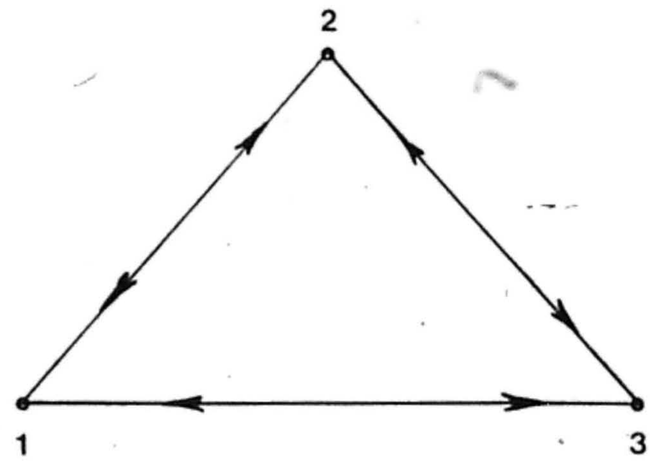
Appendix I: Digraphs of MIN



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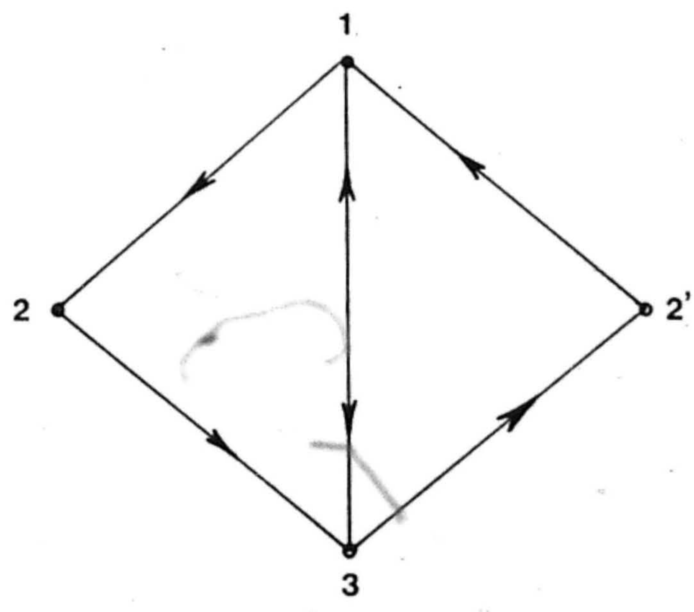

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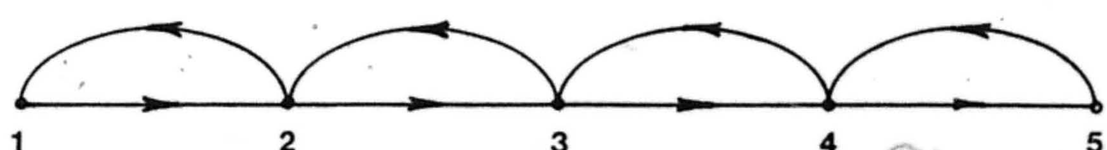

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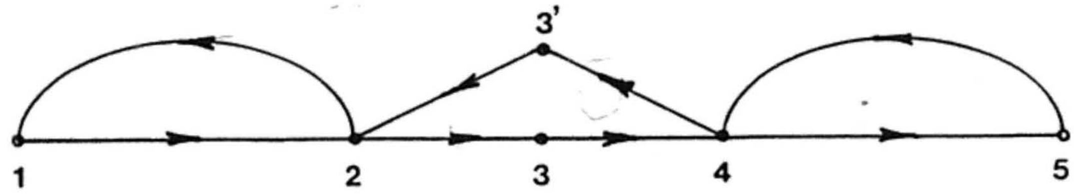

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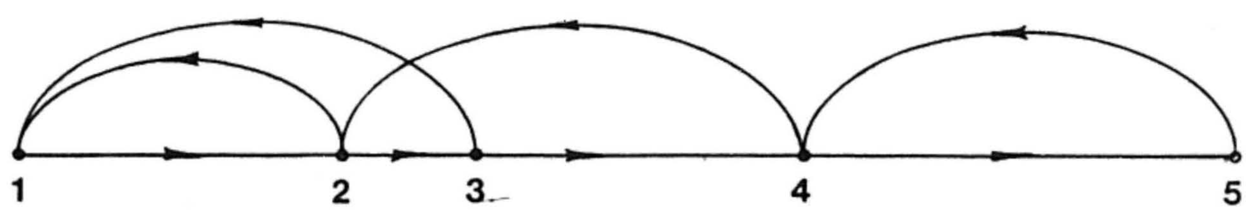

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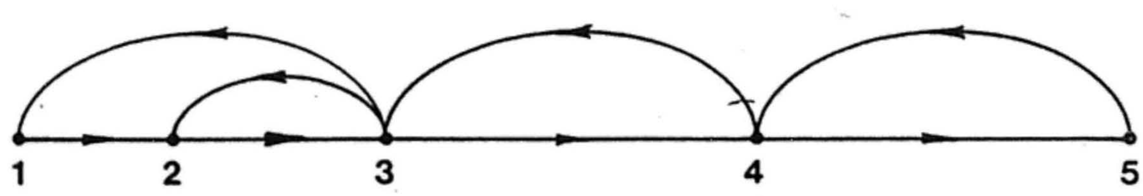

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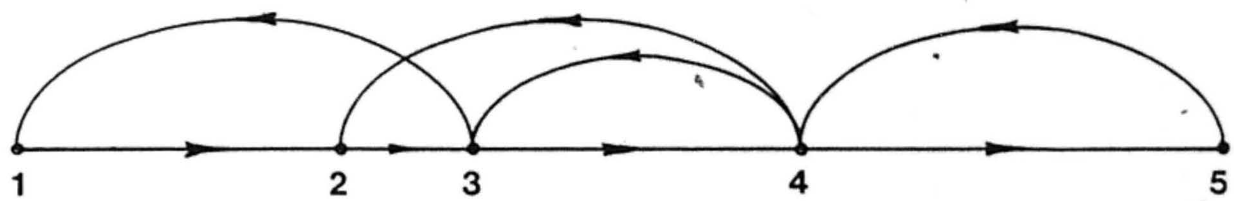

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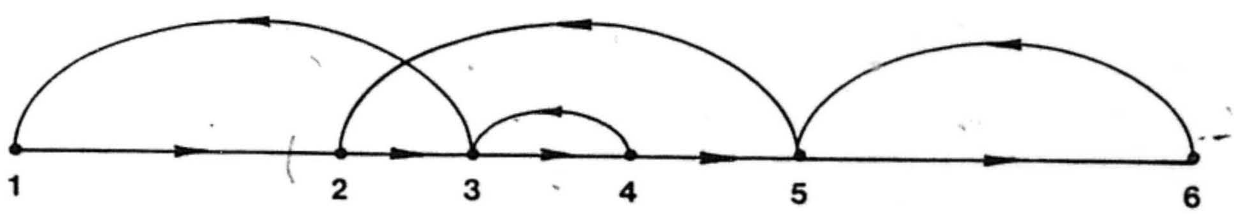

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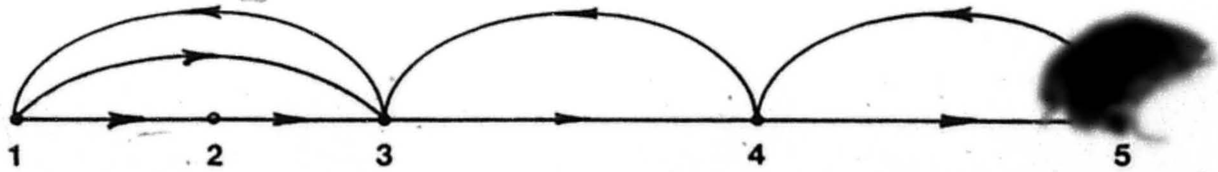

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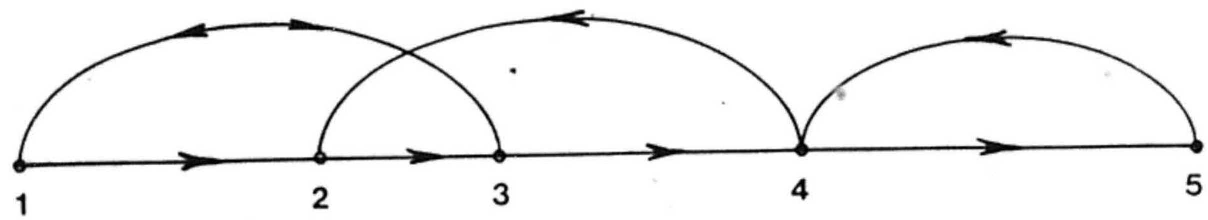

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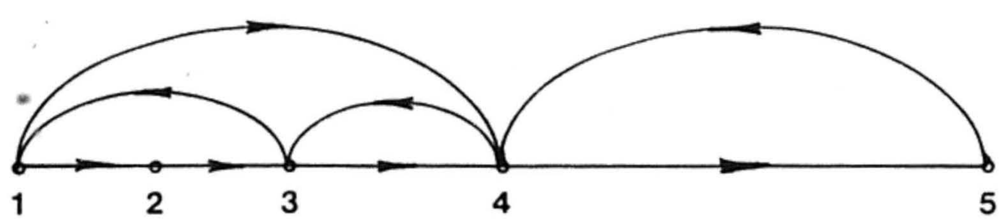

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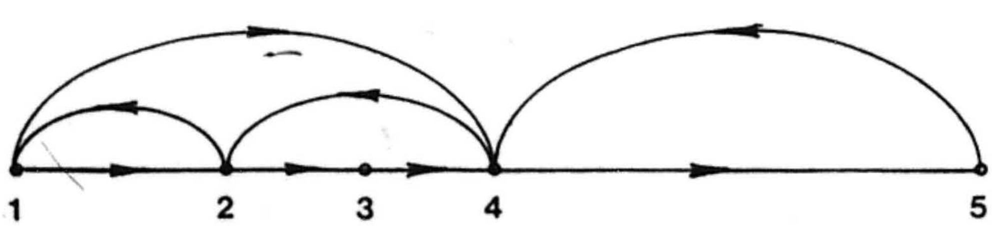

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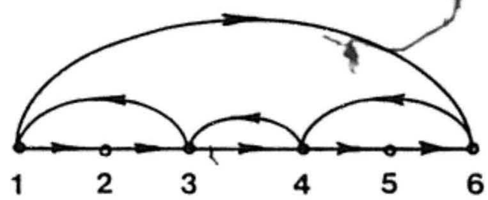

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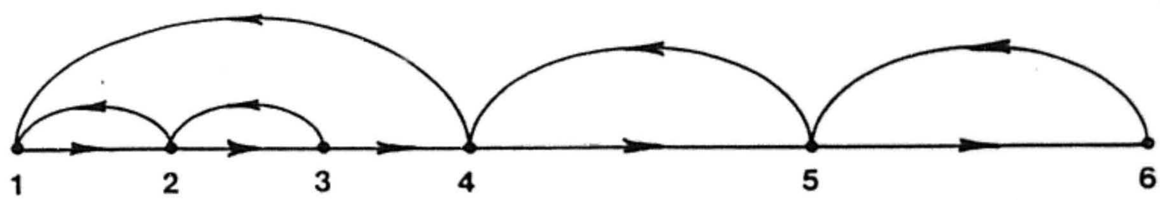

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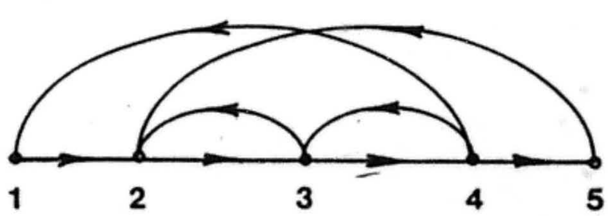

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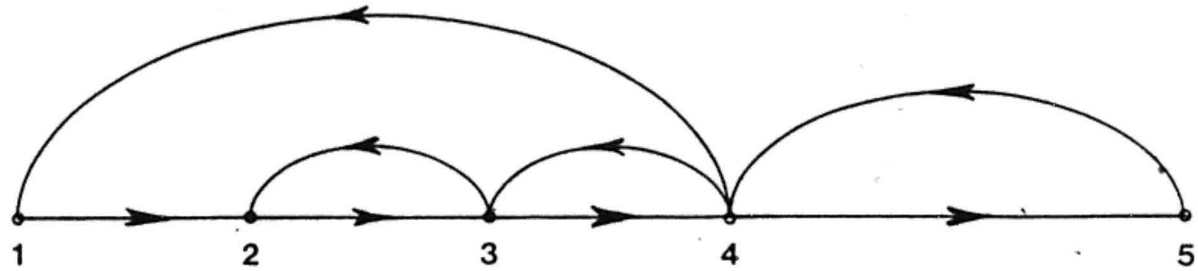

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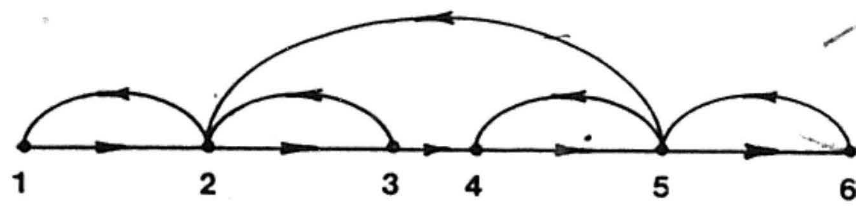

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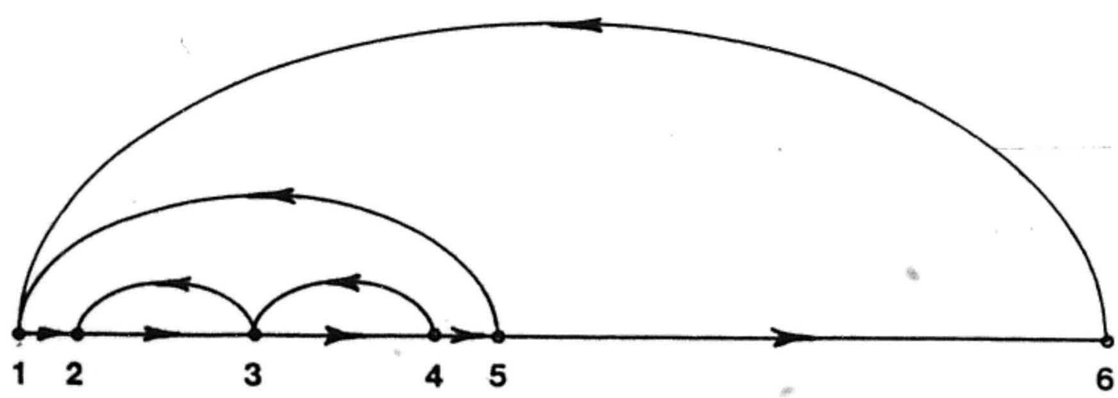

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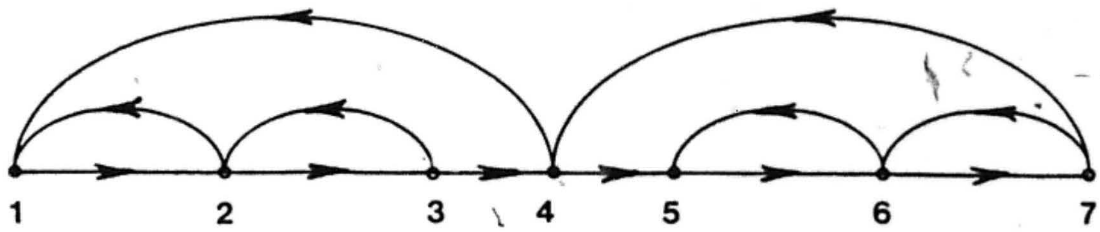

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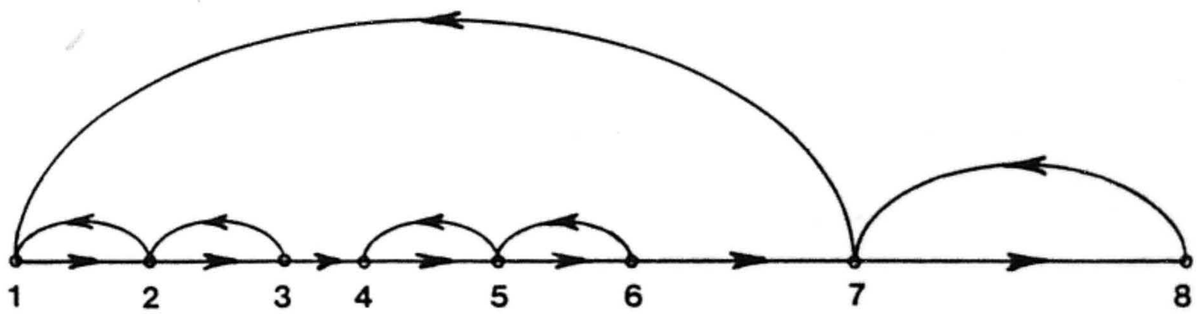

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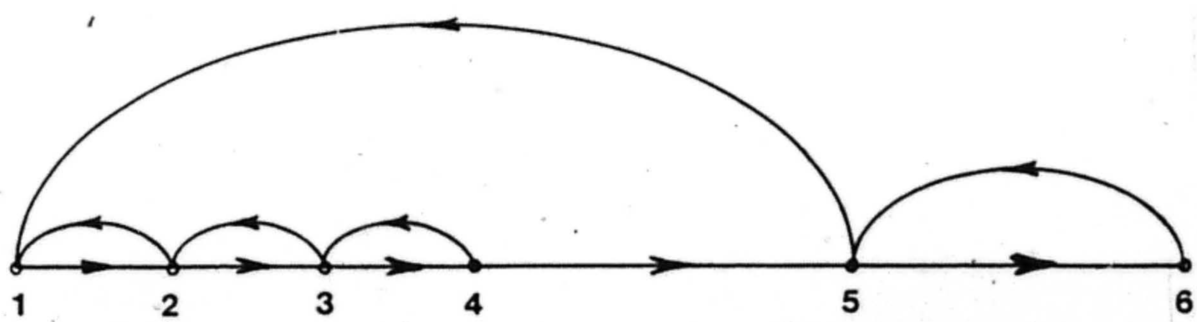

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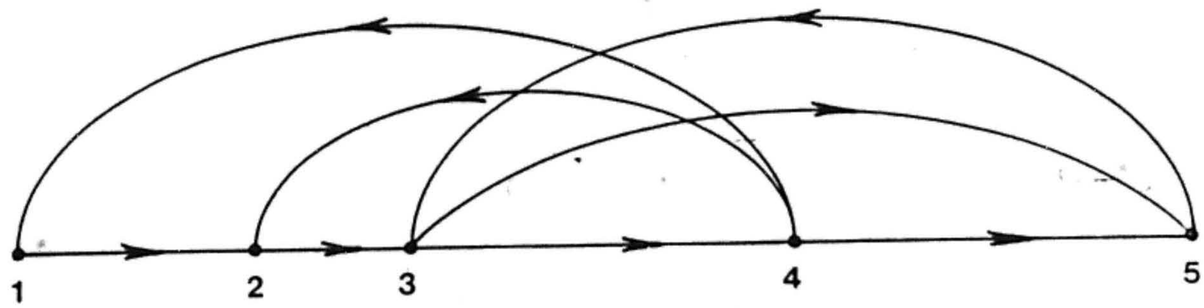

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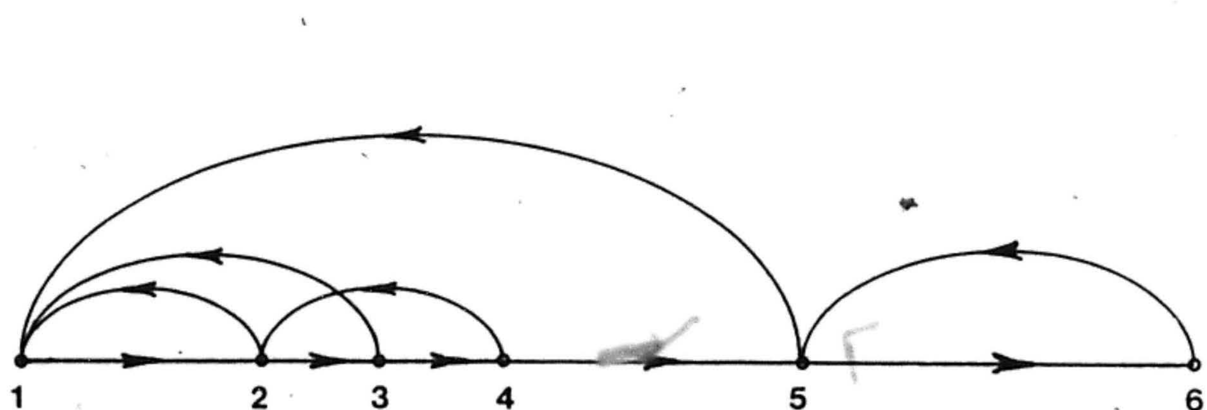

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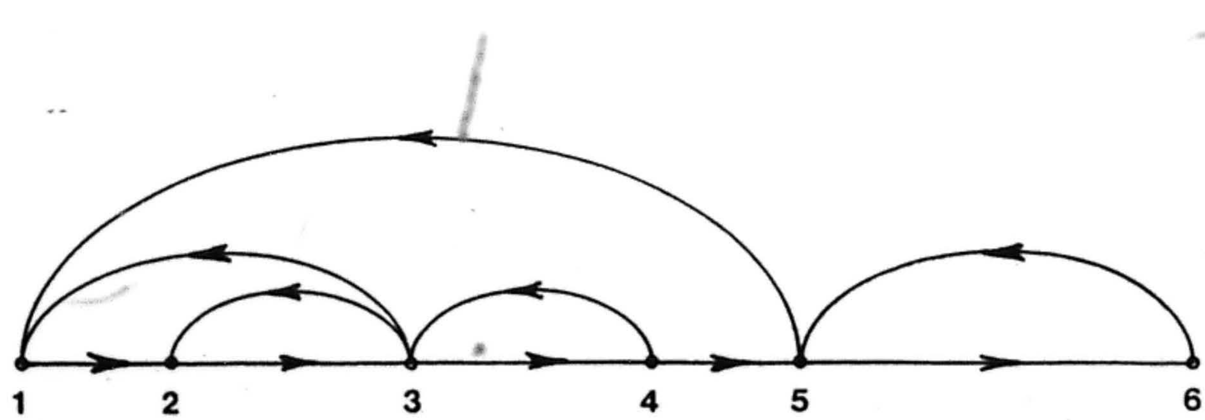

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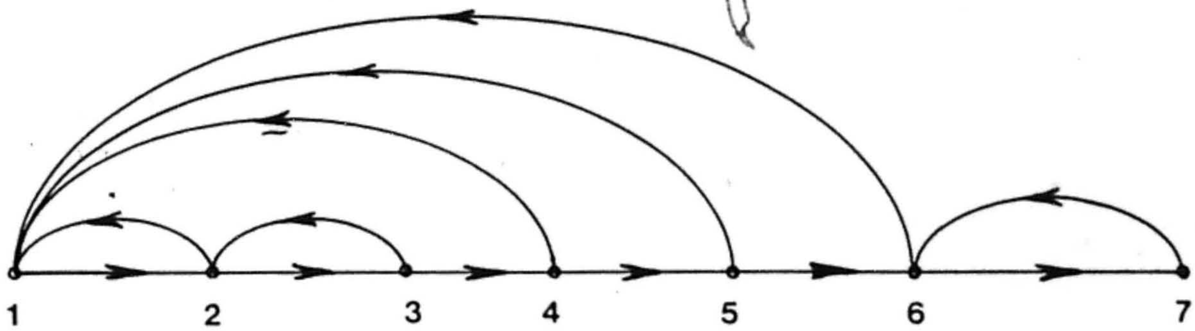

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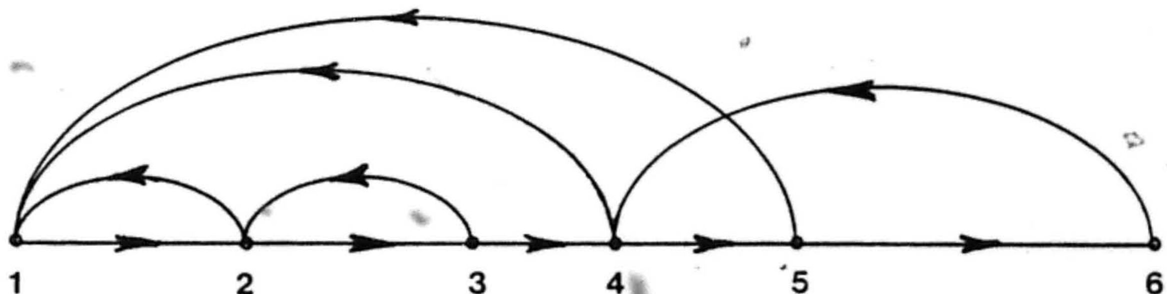

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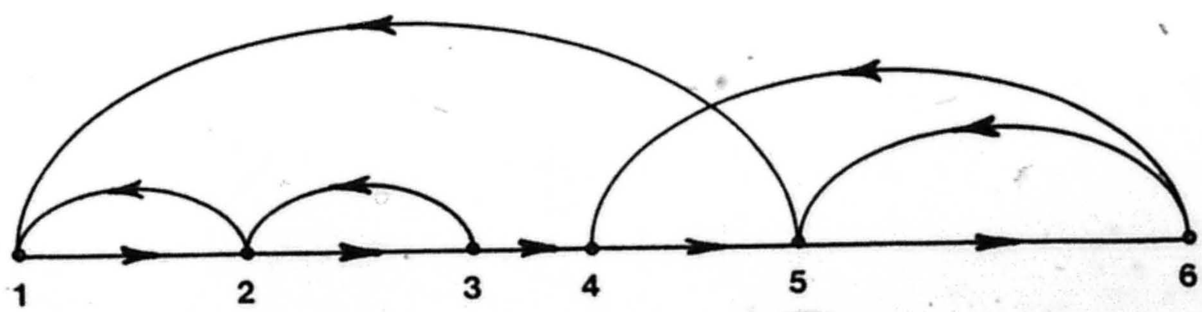

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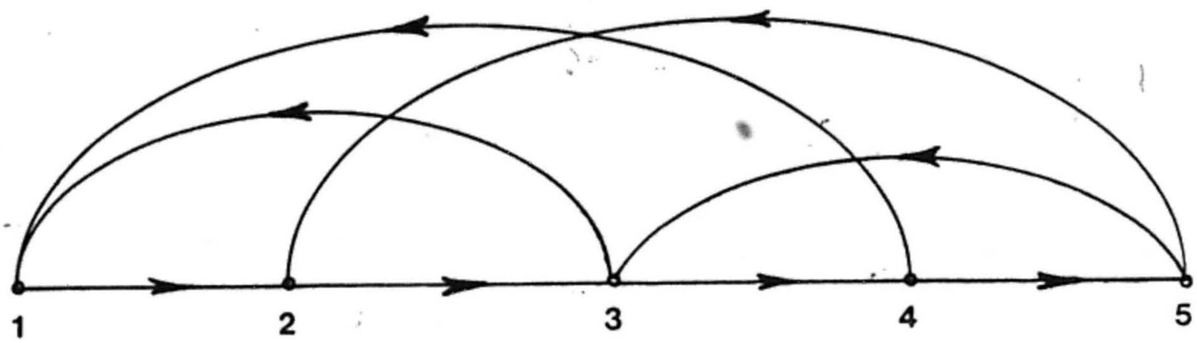

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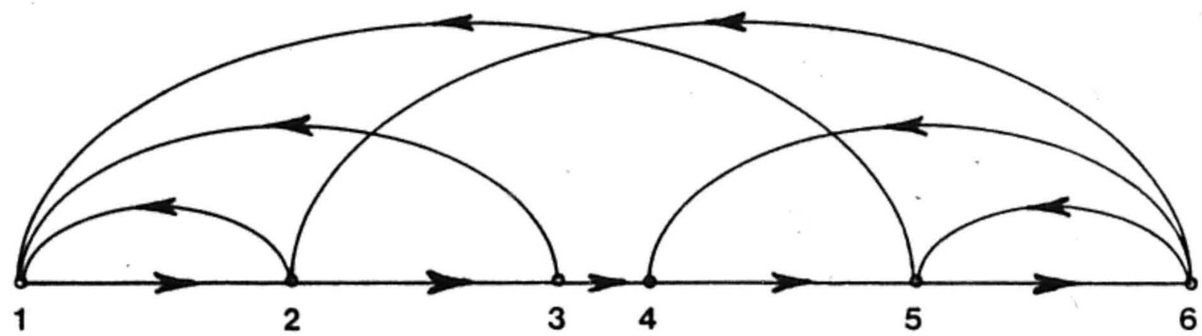

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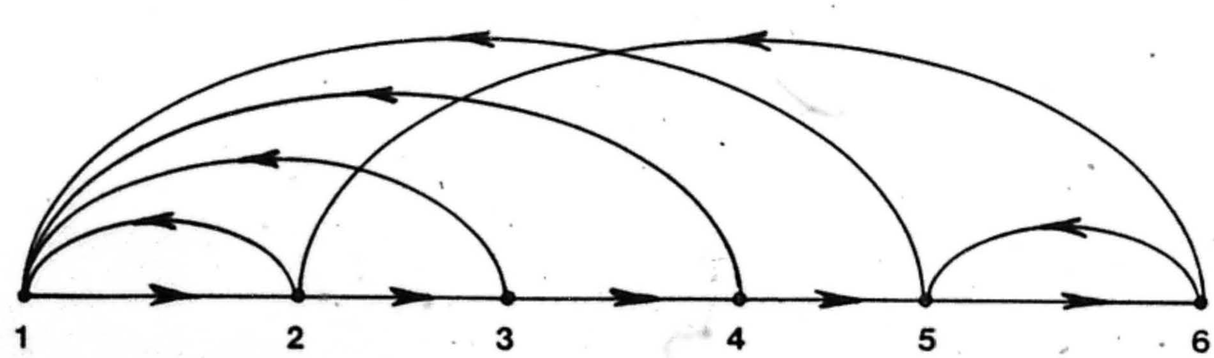

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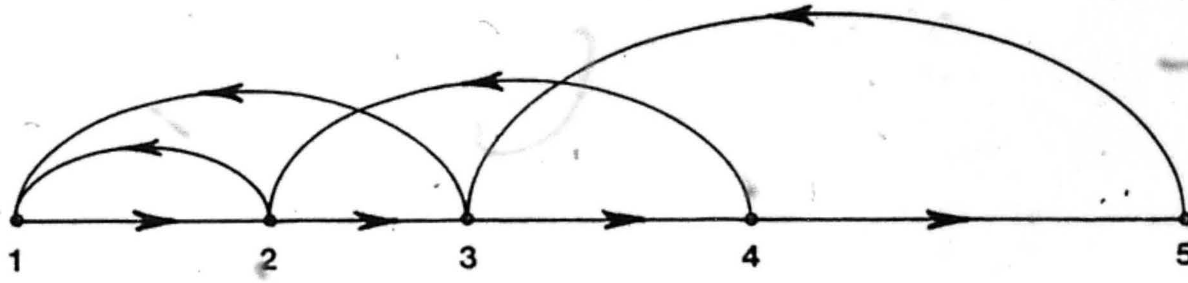

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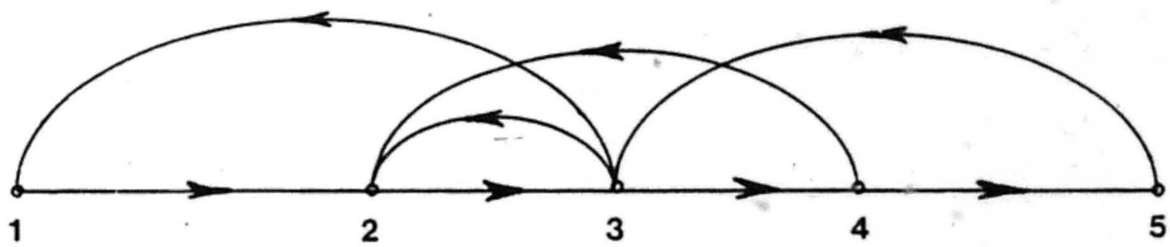

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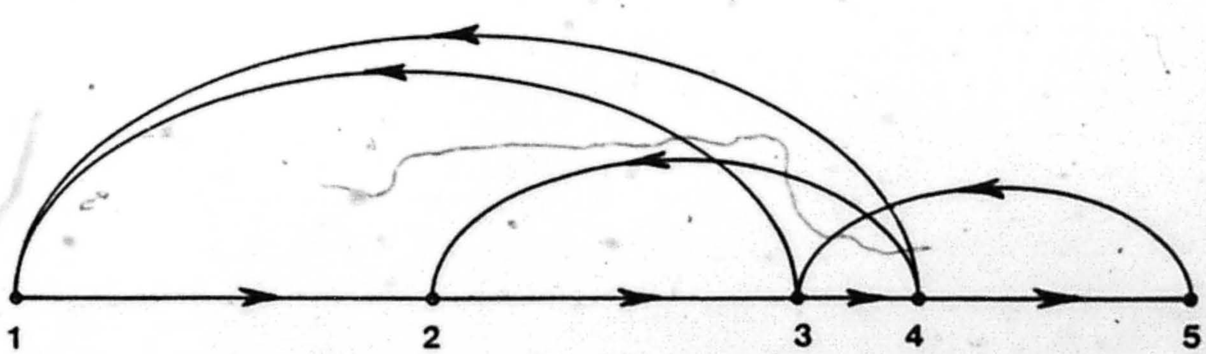

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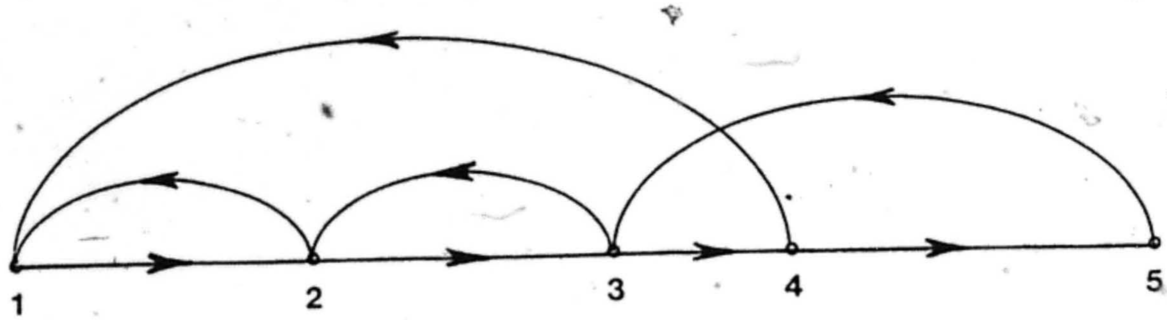

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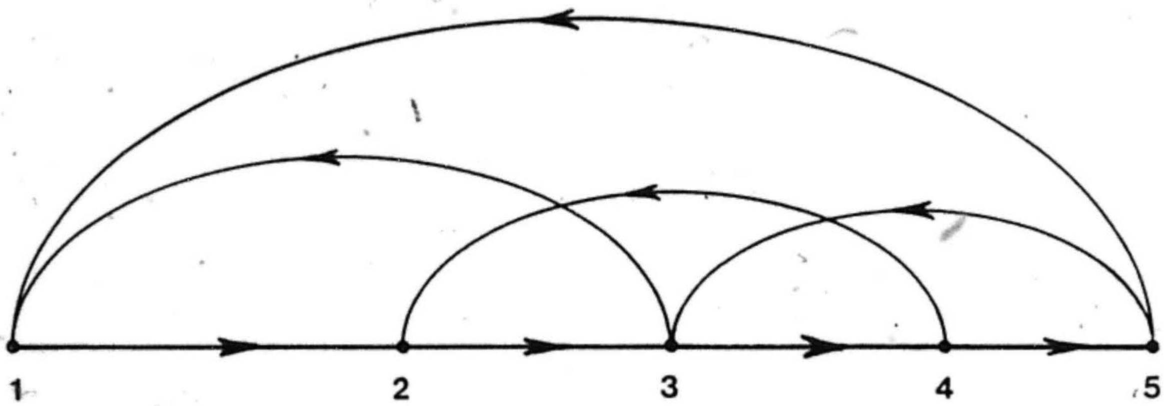

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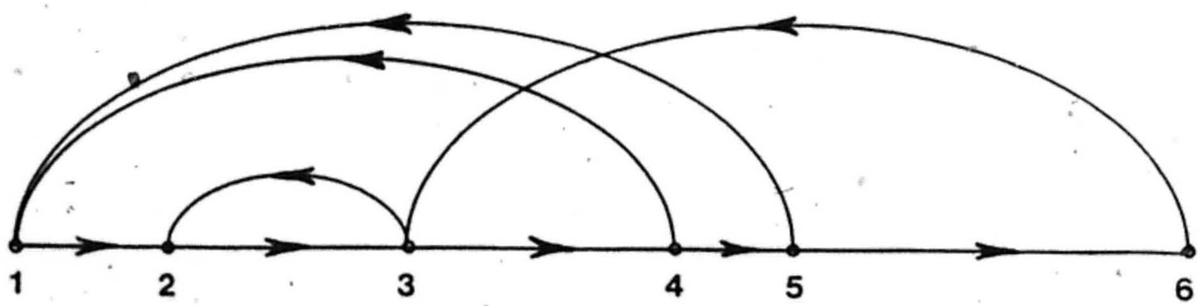

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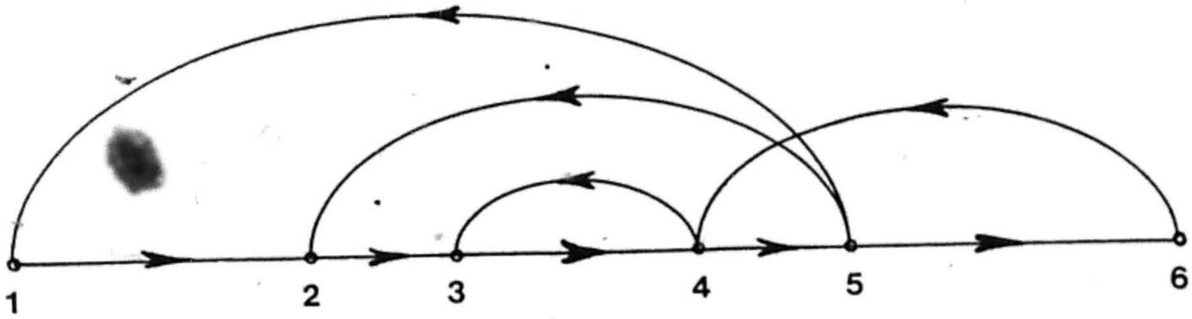

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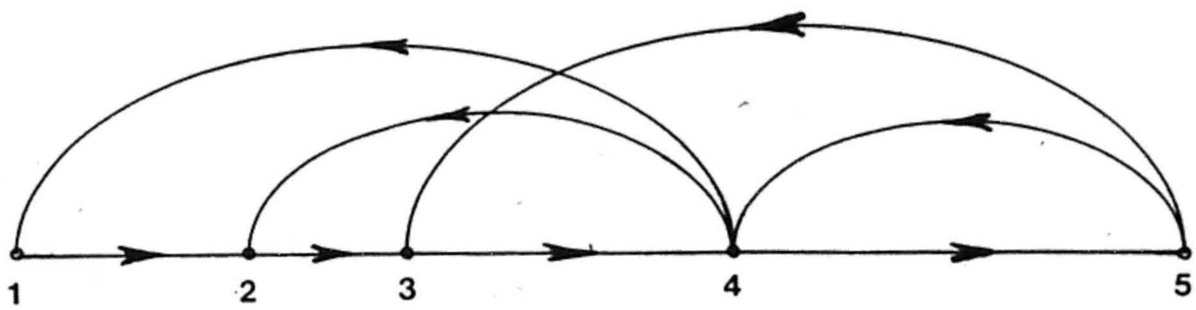

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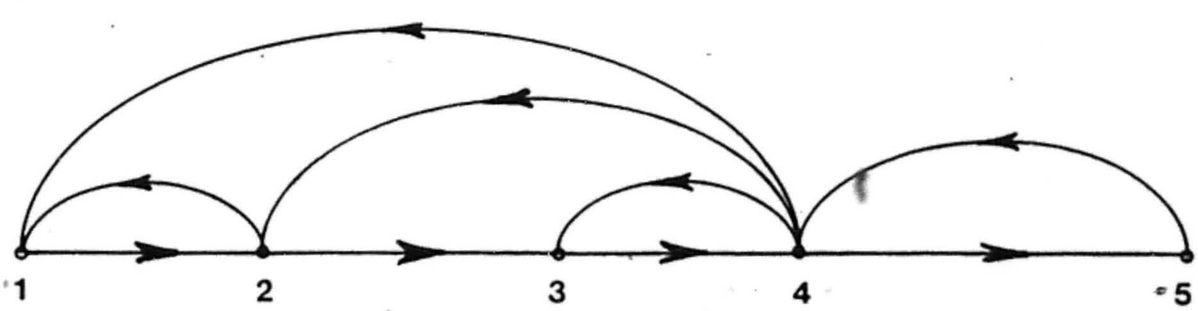

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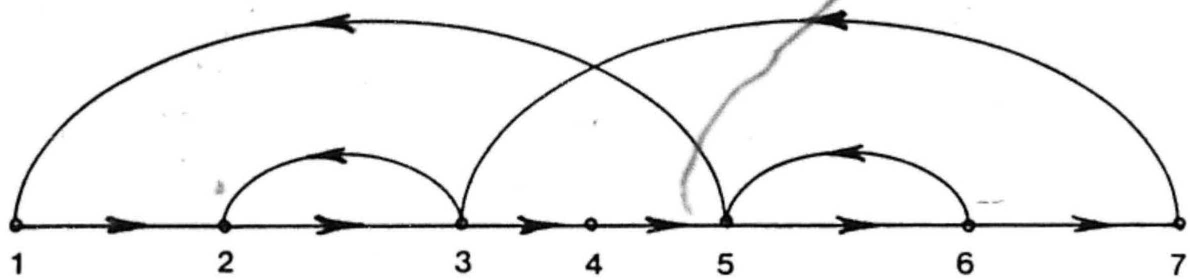

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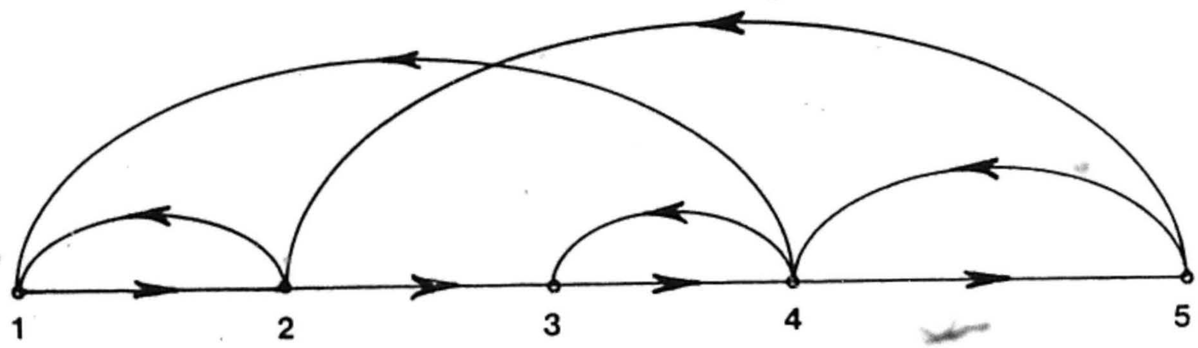

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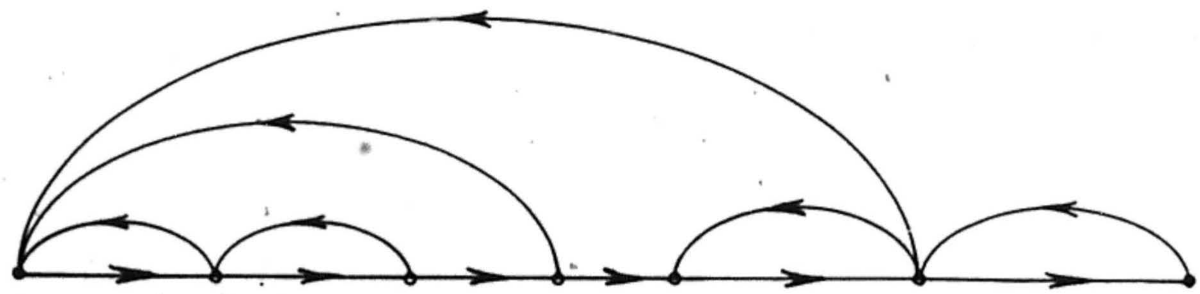

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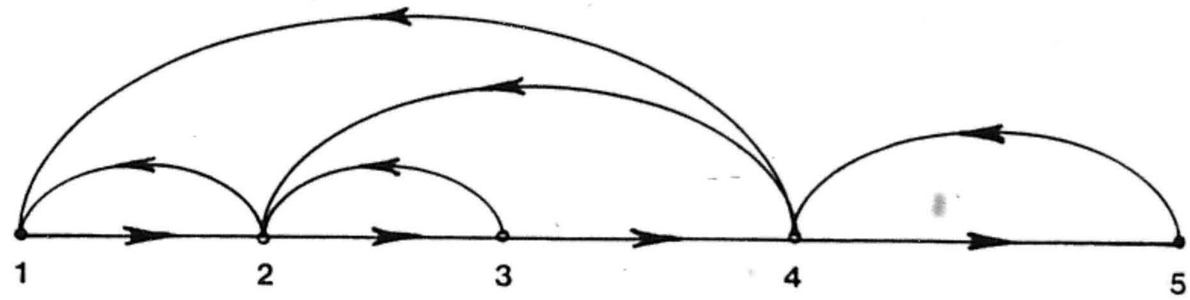

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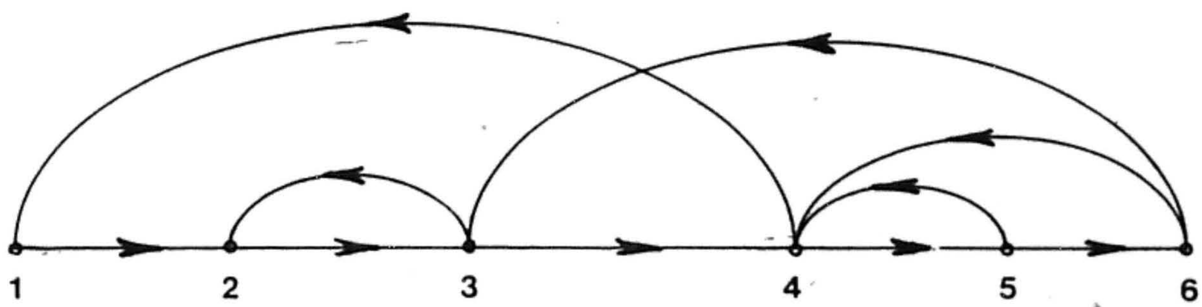

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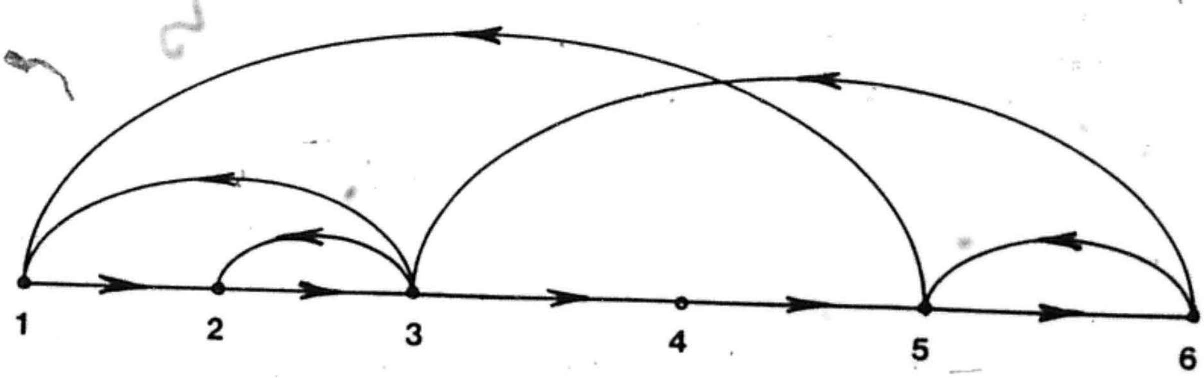

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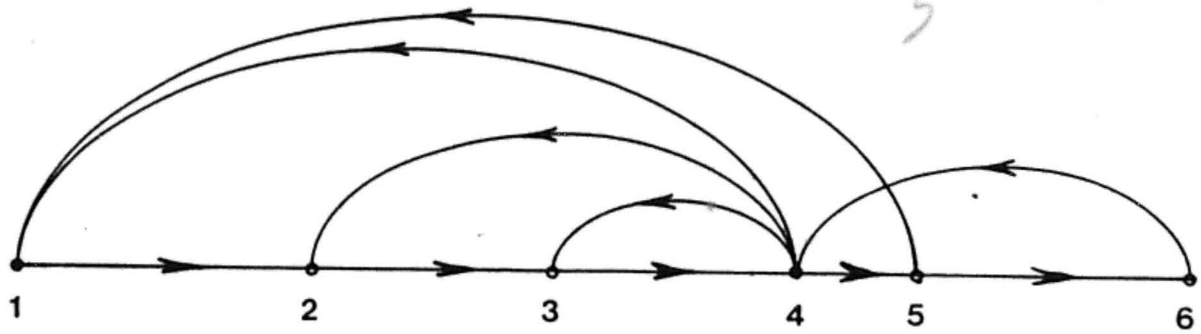

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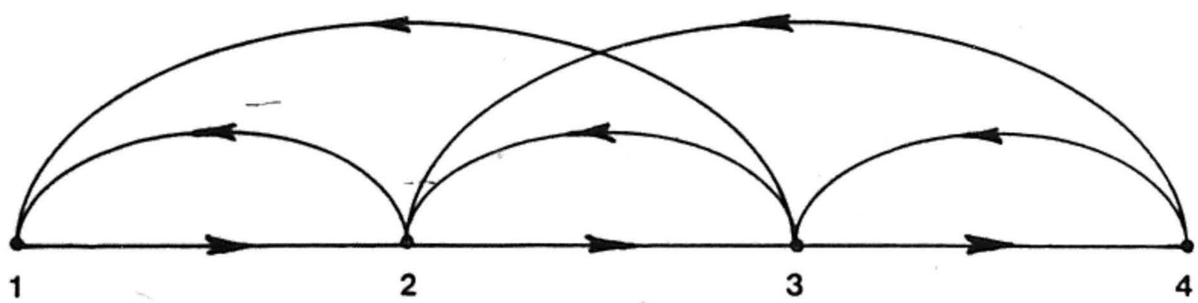

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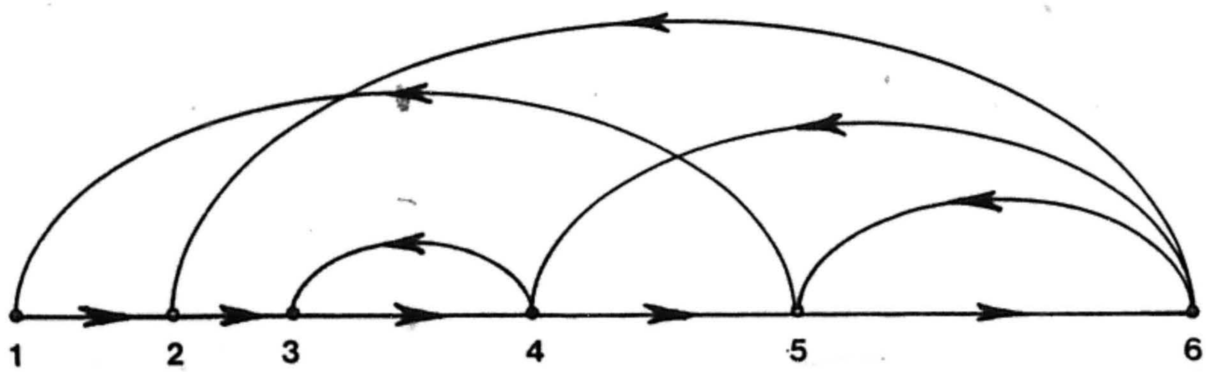

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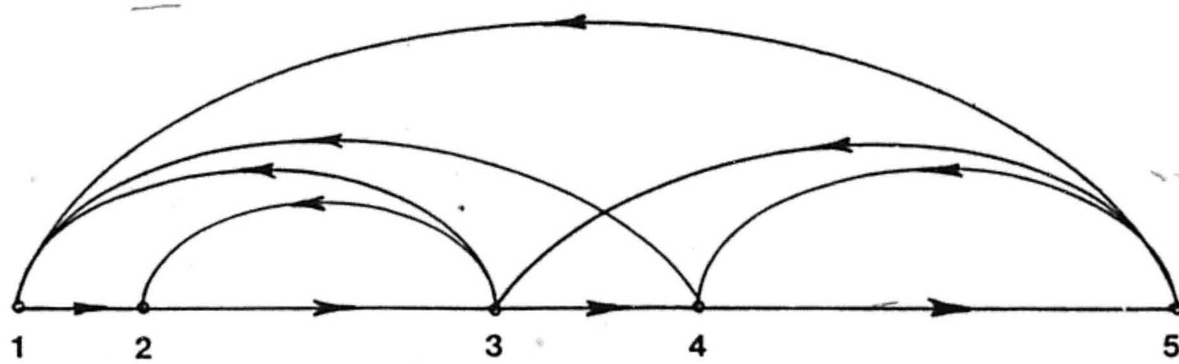

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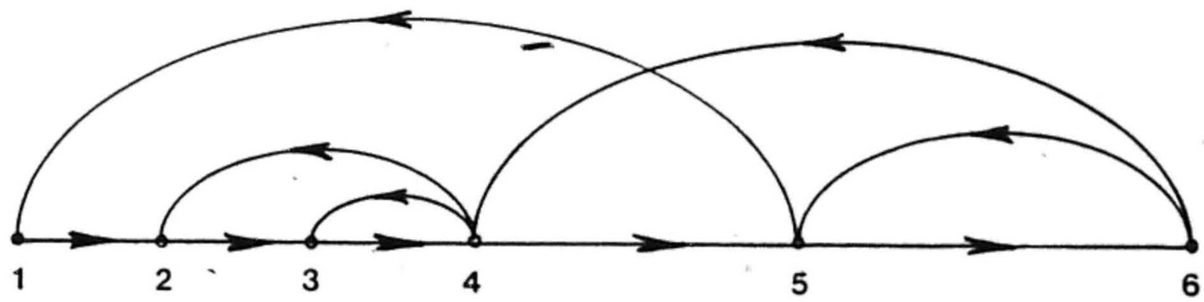

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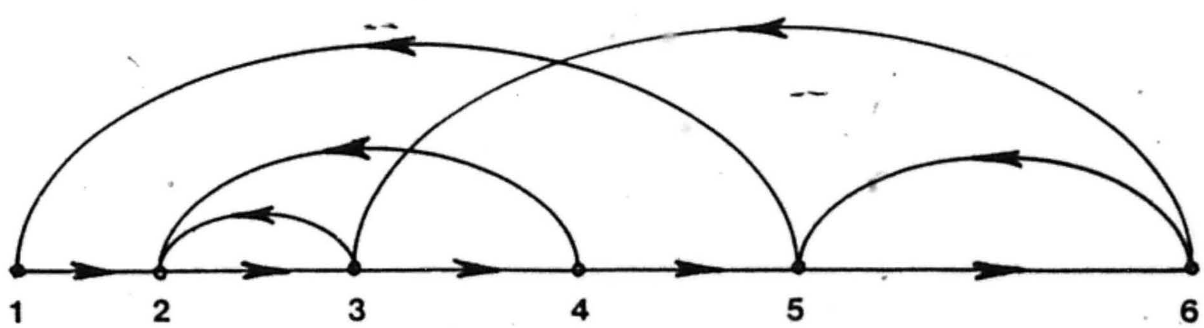

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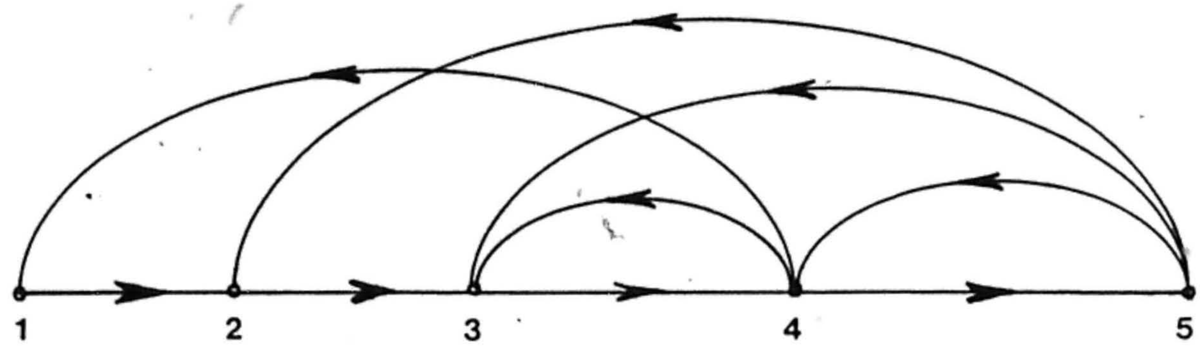

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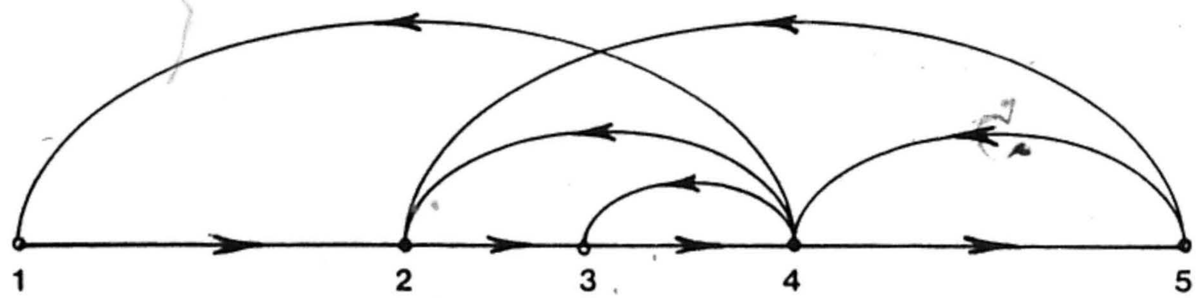

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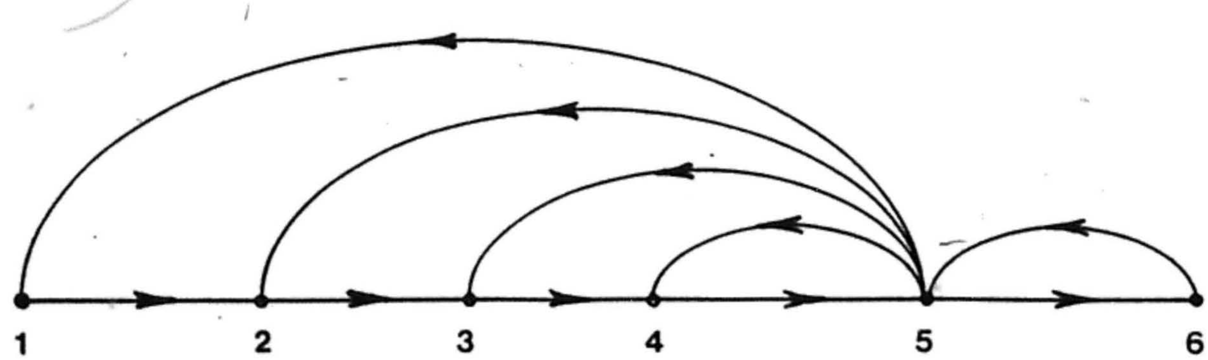

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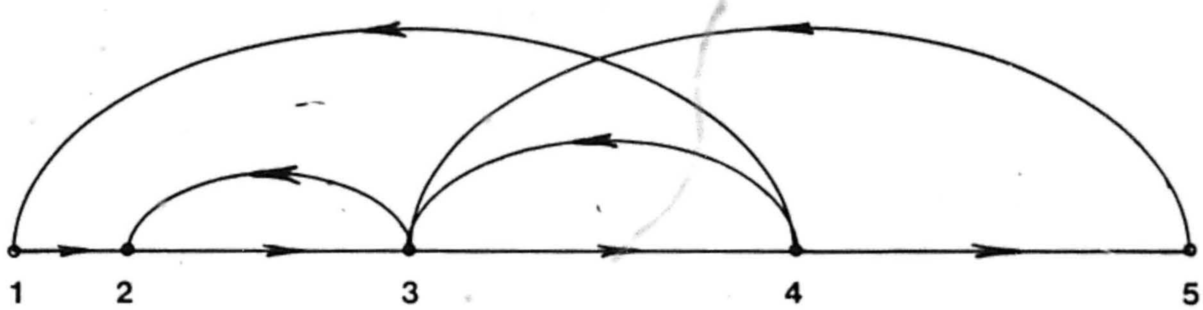

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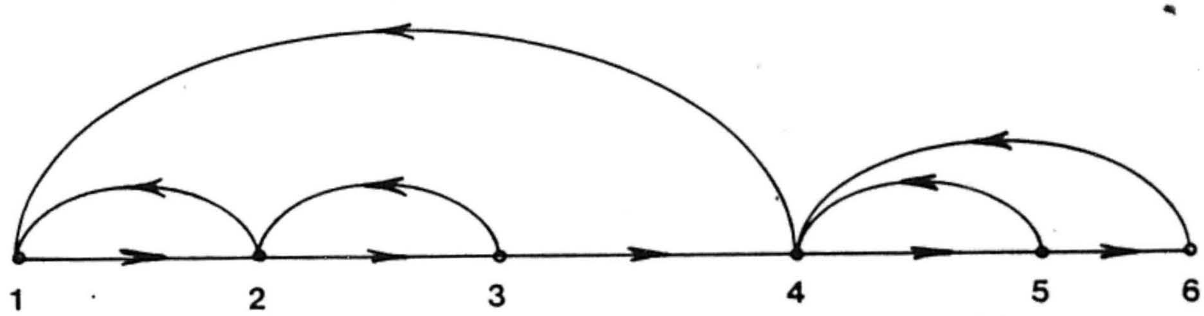

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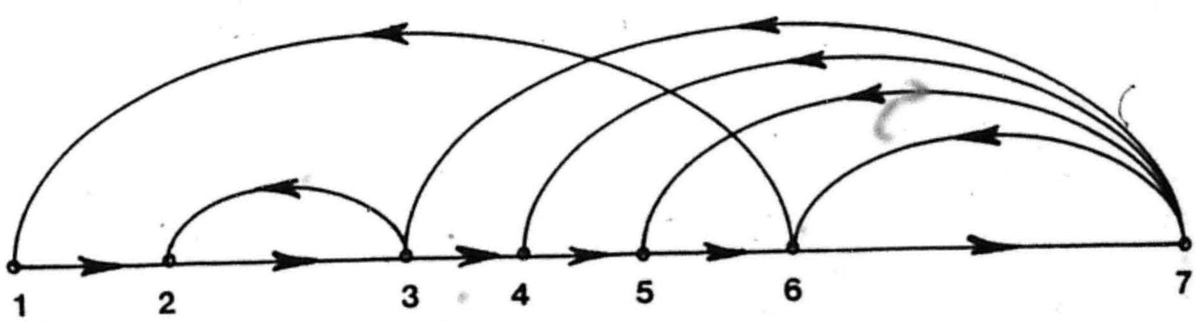

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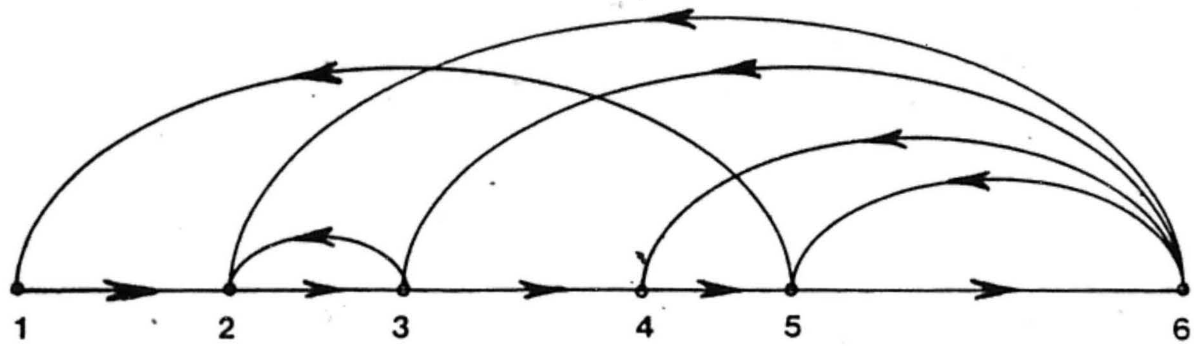

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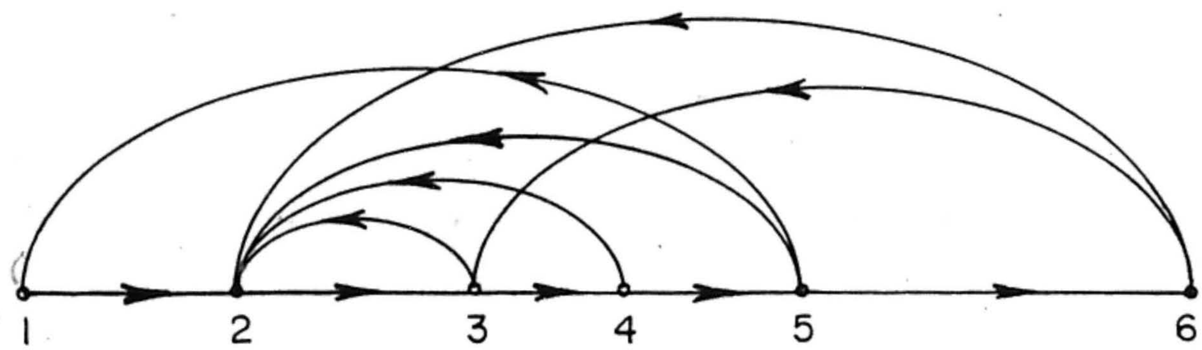

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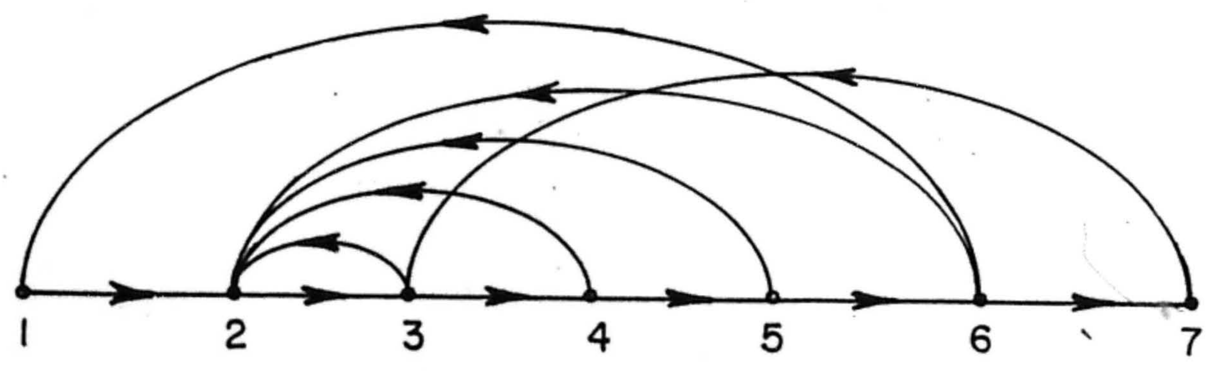

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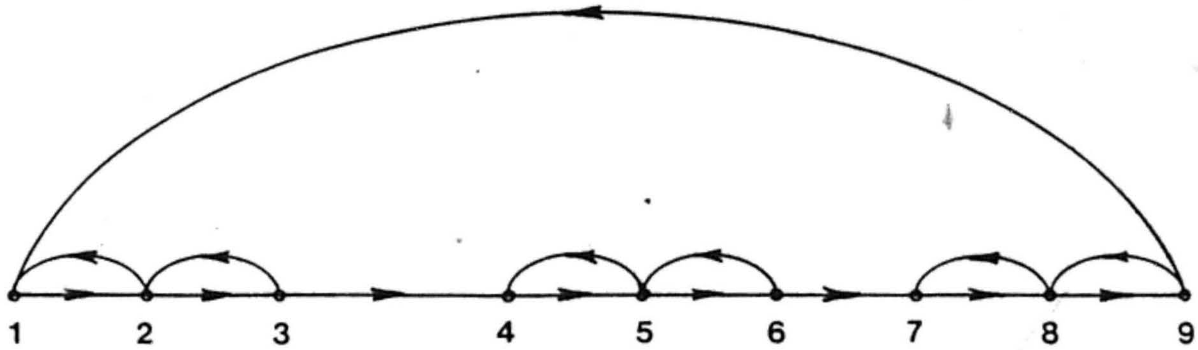

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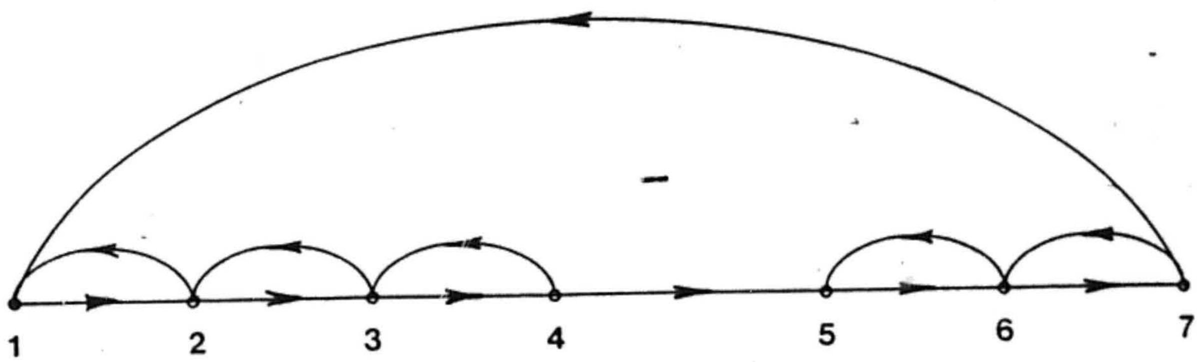

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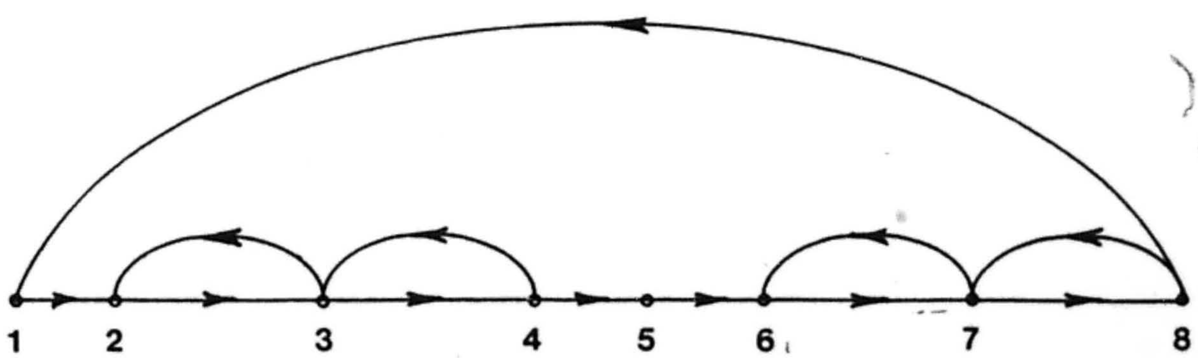

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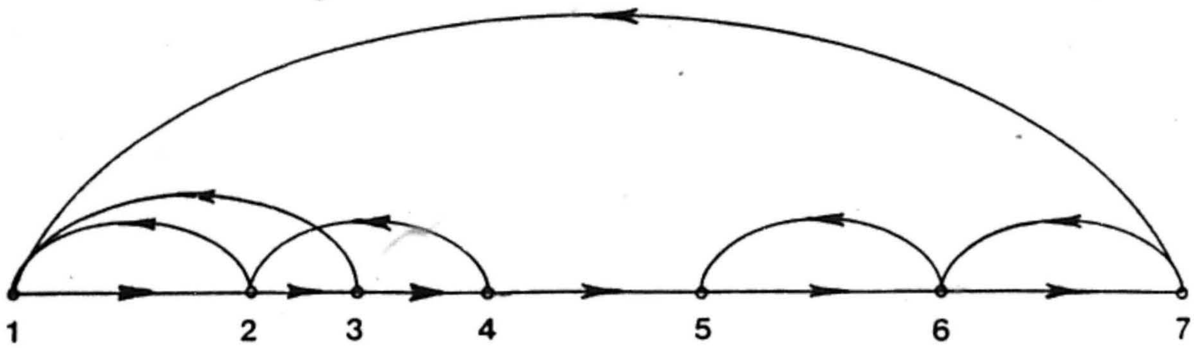

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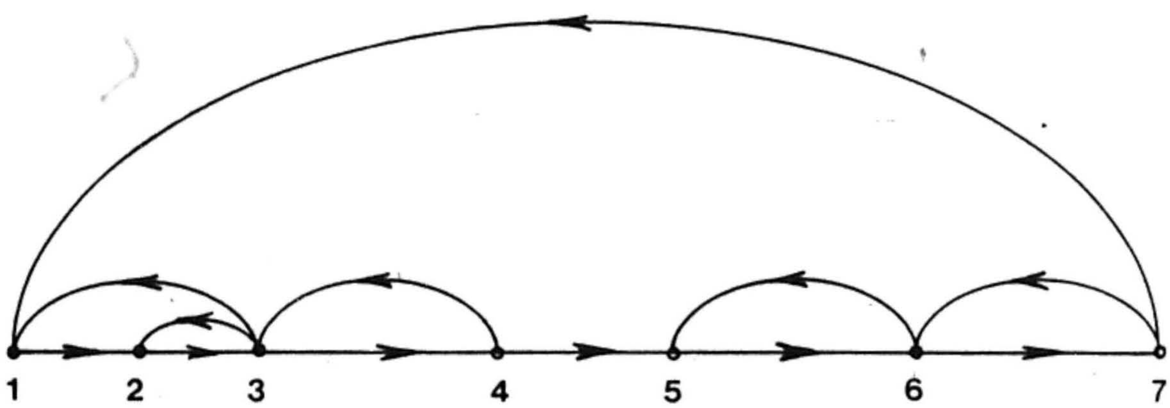

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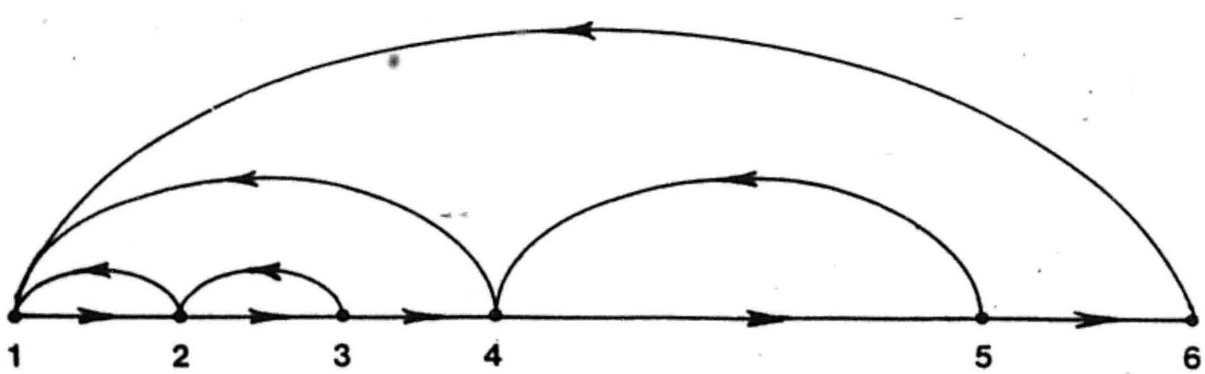

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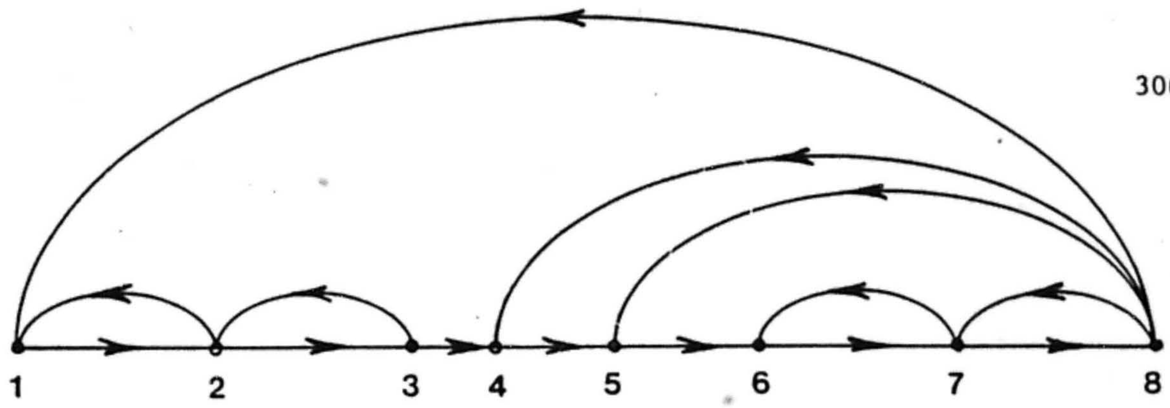

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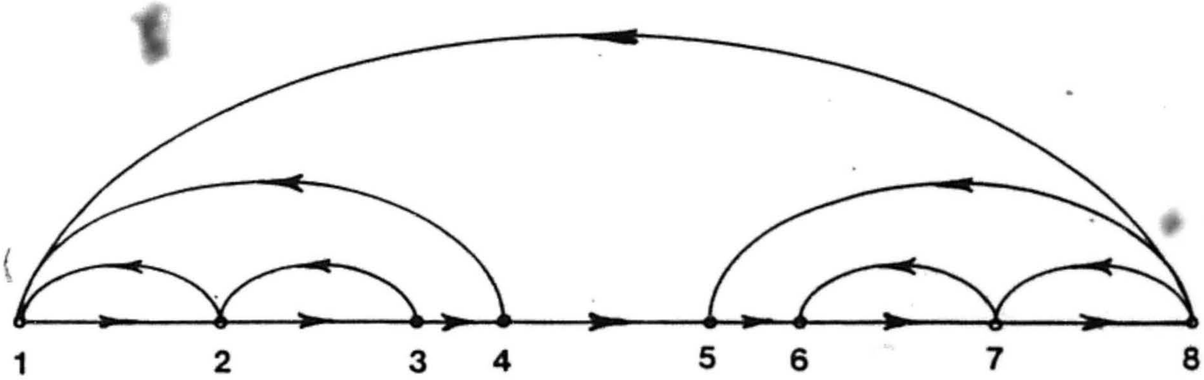

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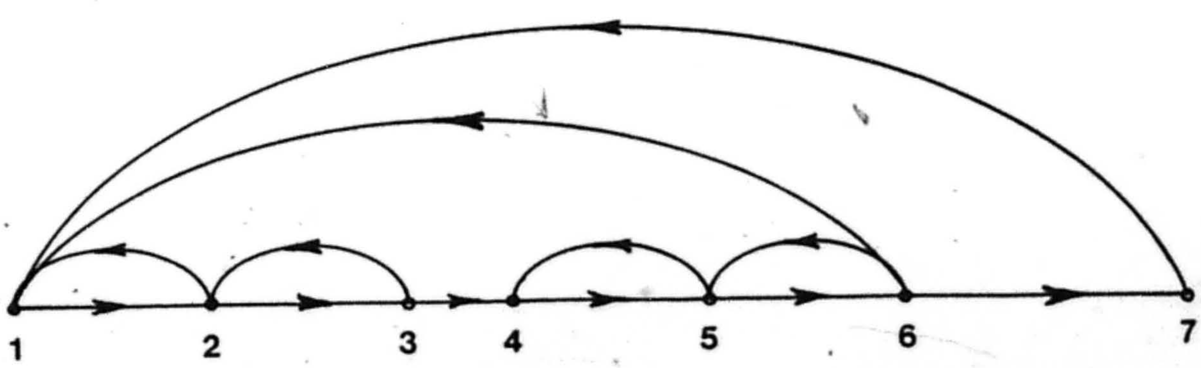

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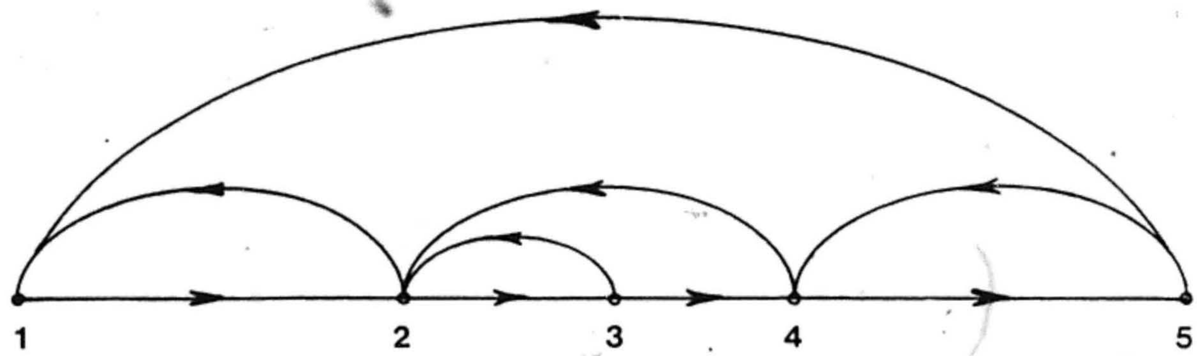

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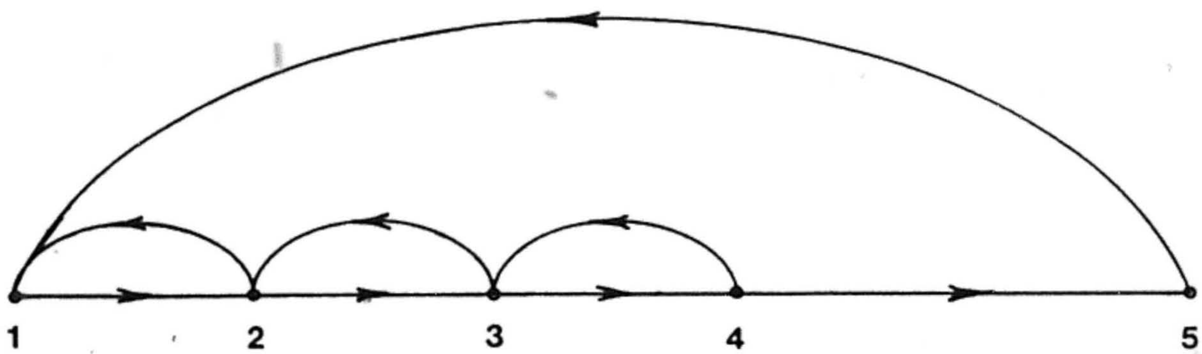

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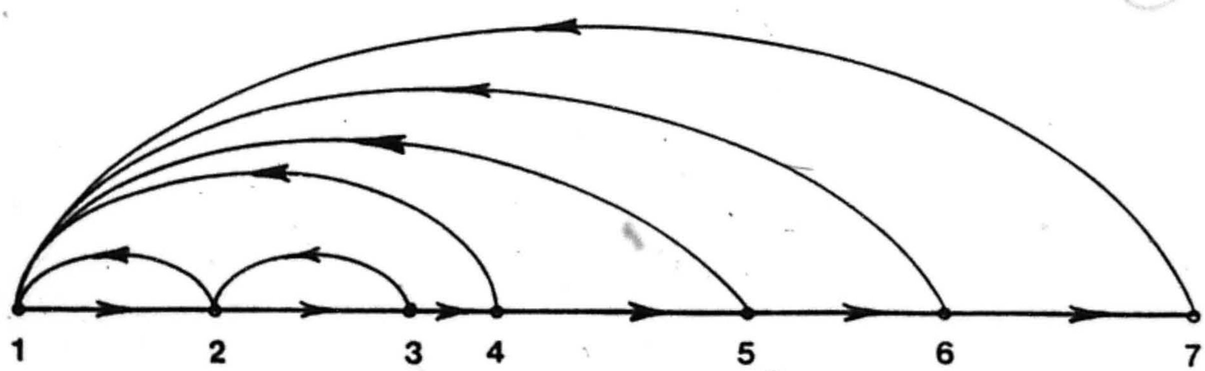

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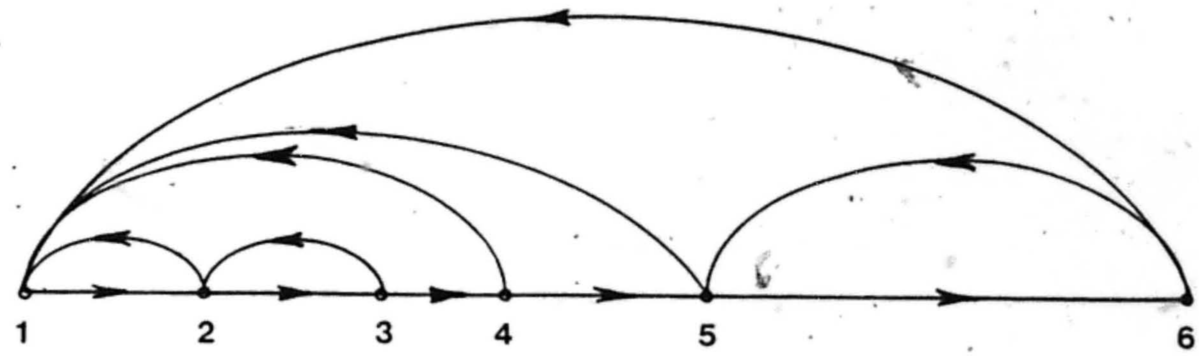

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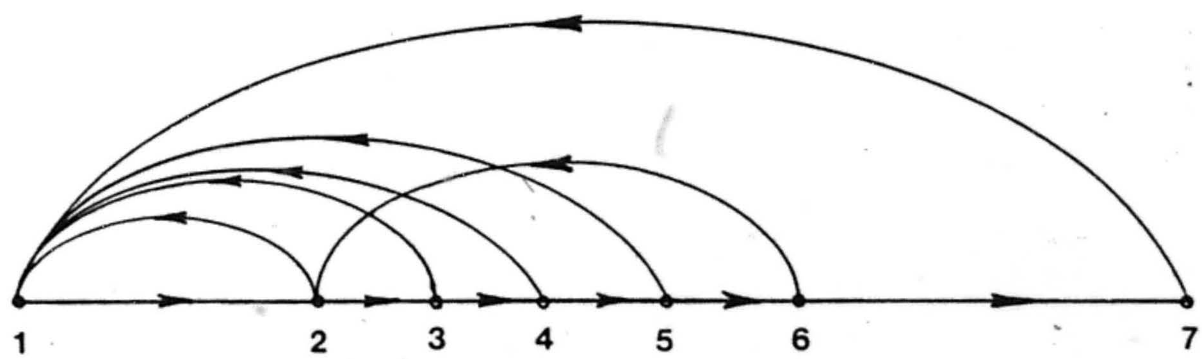

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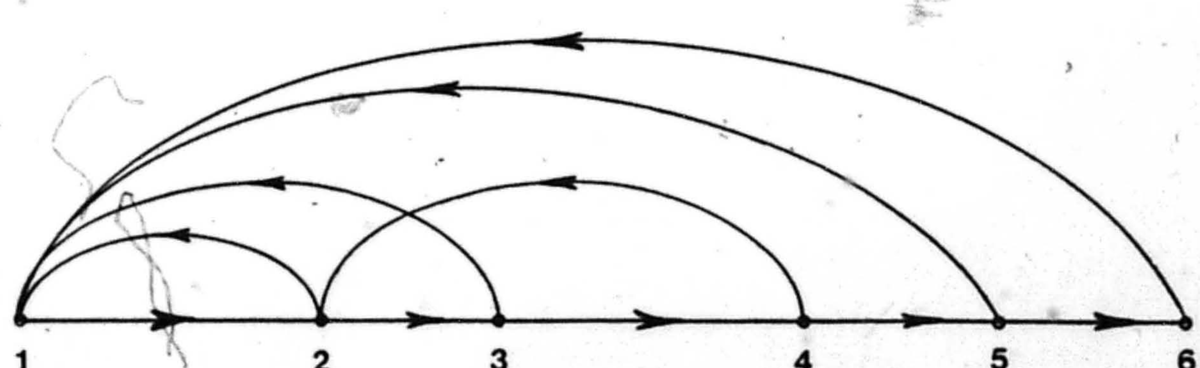

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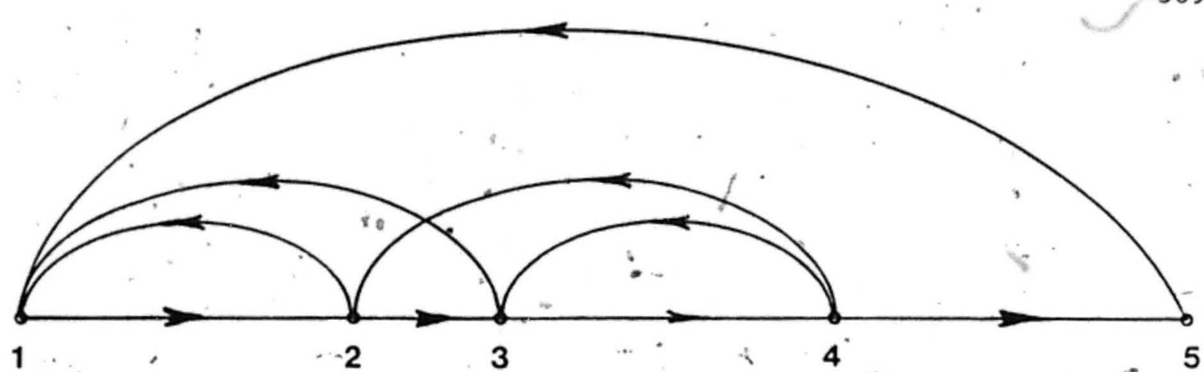

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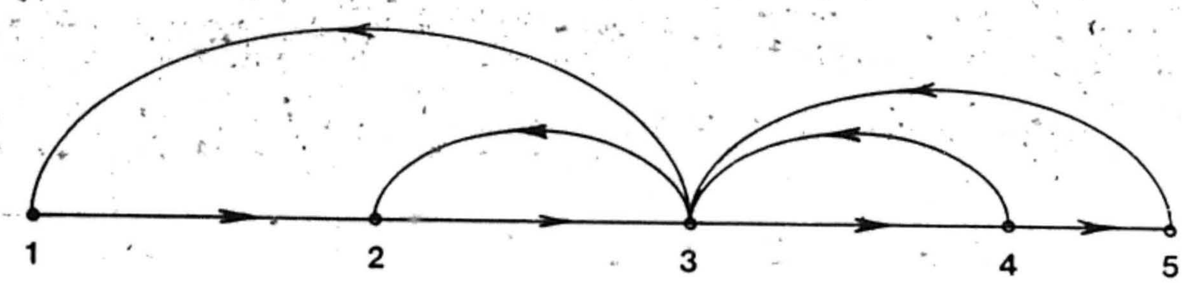

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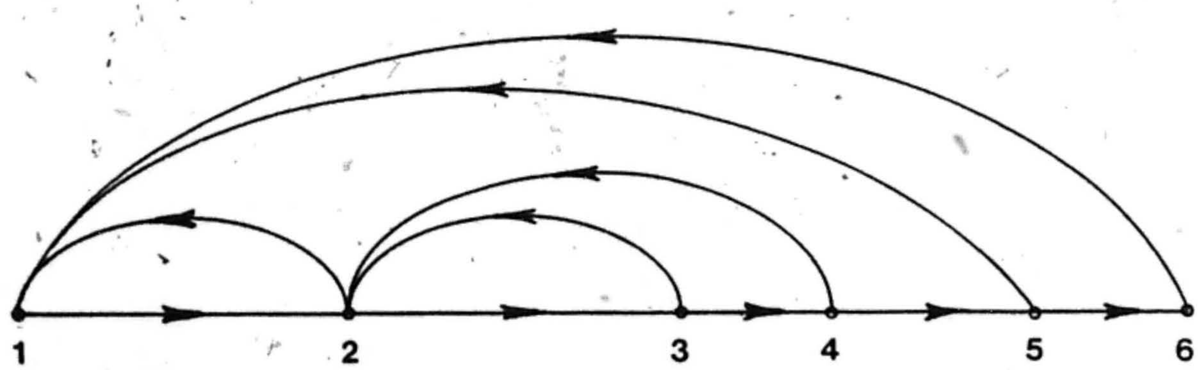

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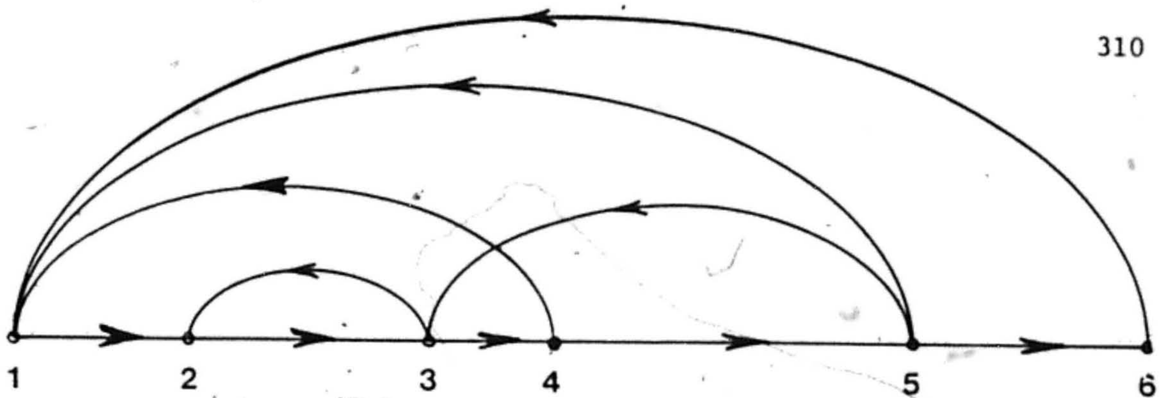

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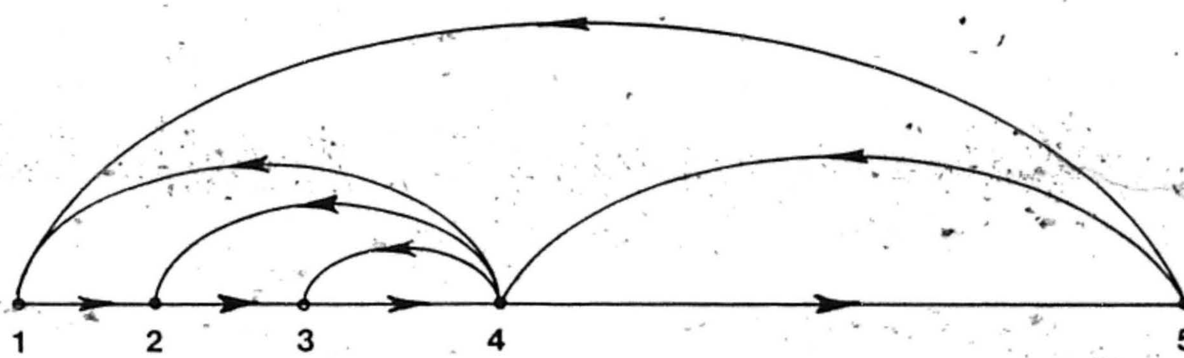

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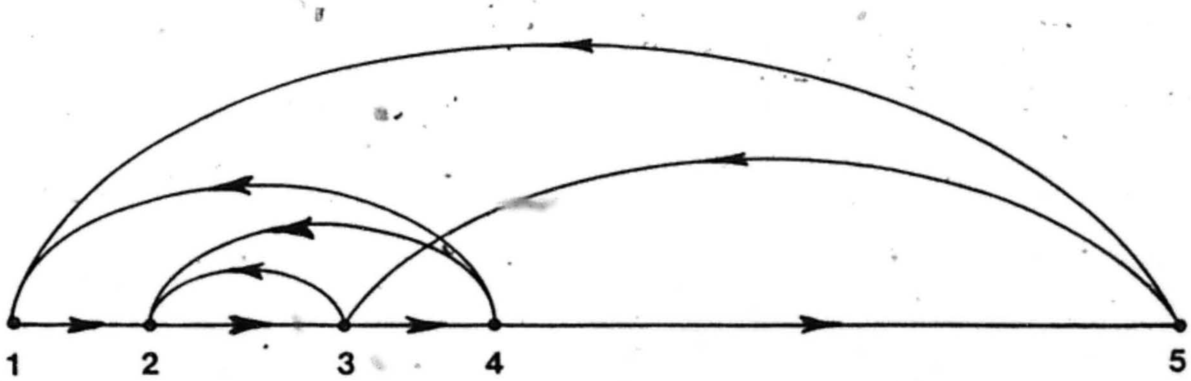

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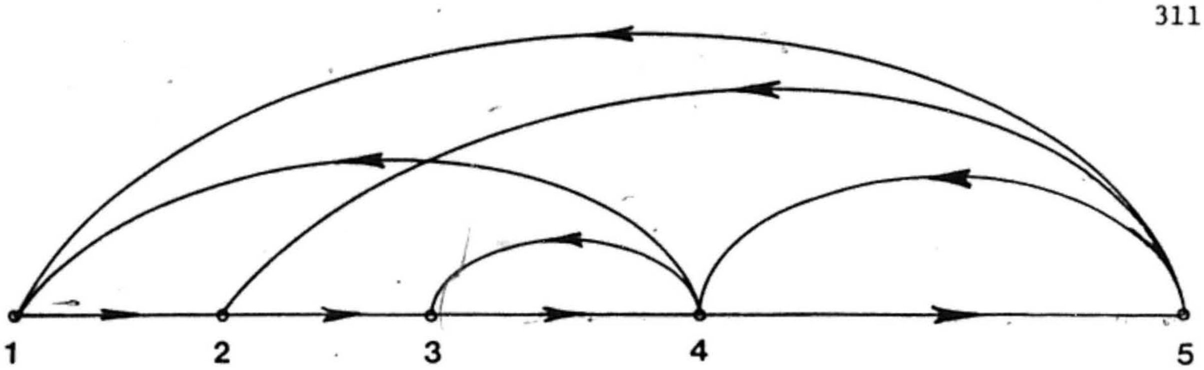

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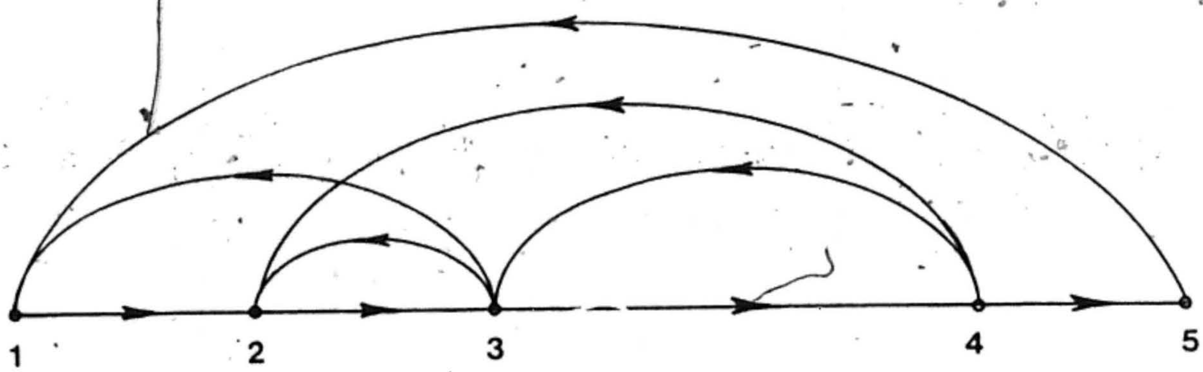

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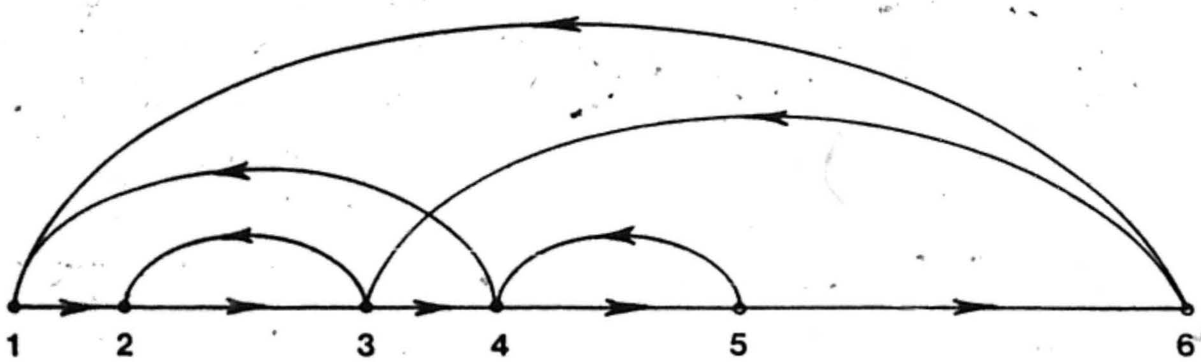

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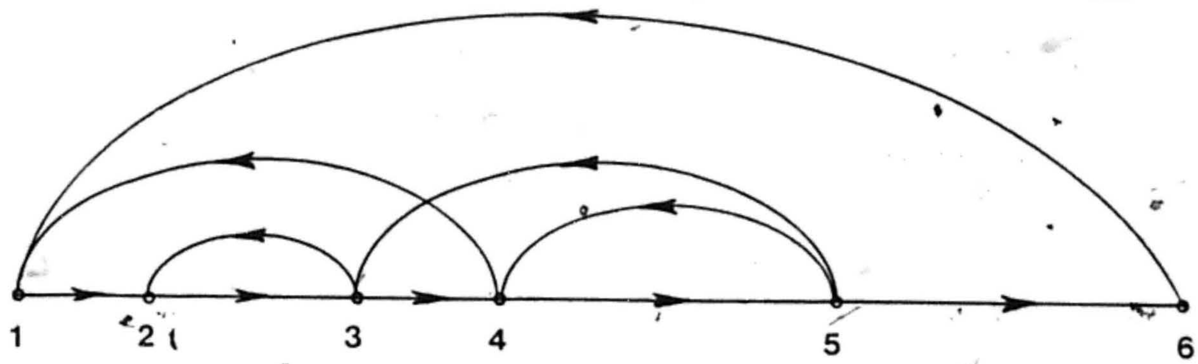

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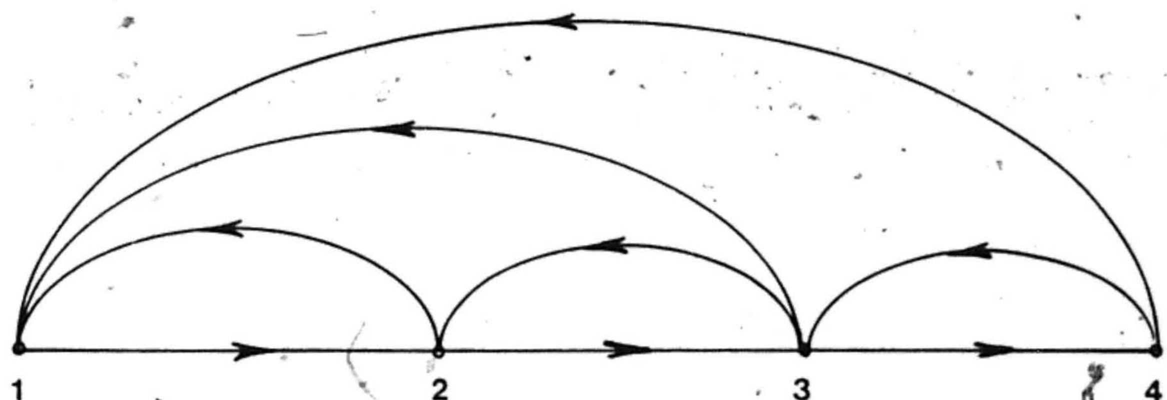

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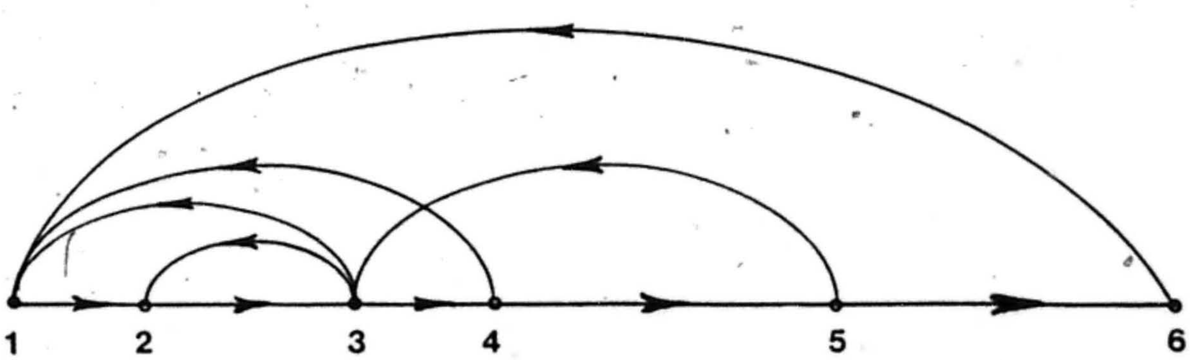

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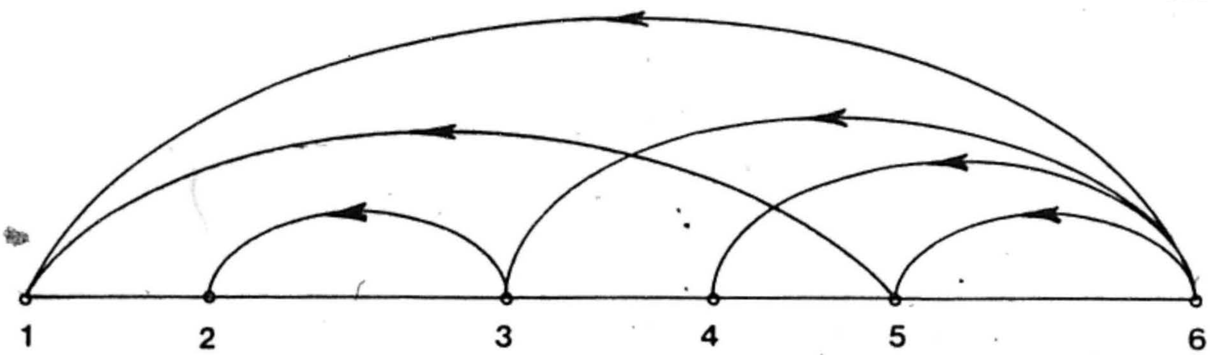

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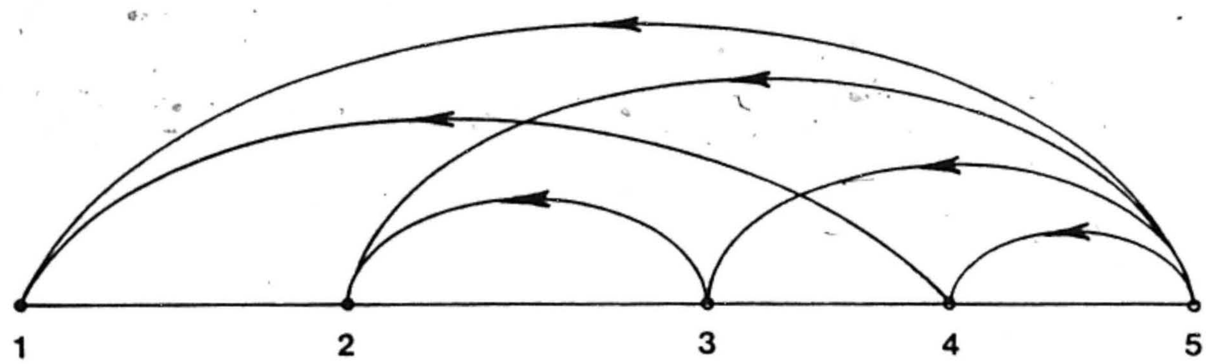

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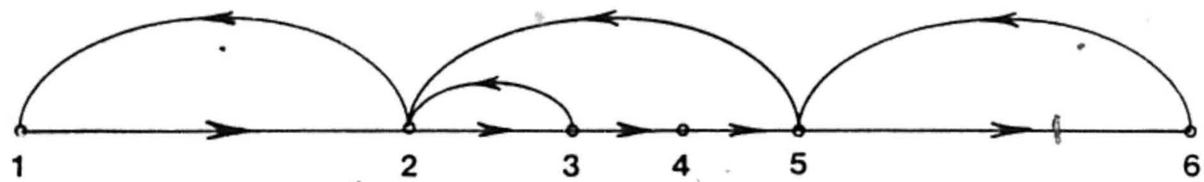
Appendix II: Digraphs of MAX



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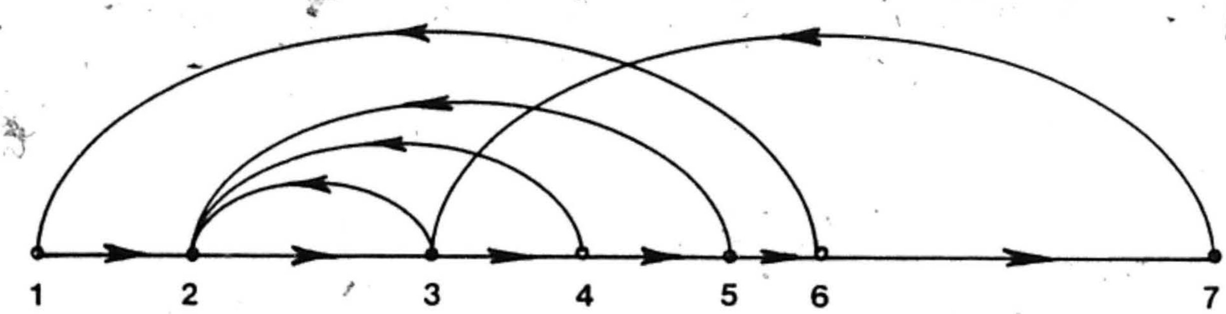

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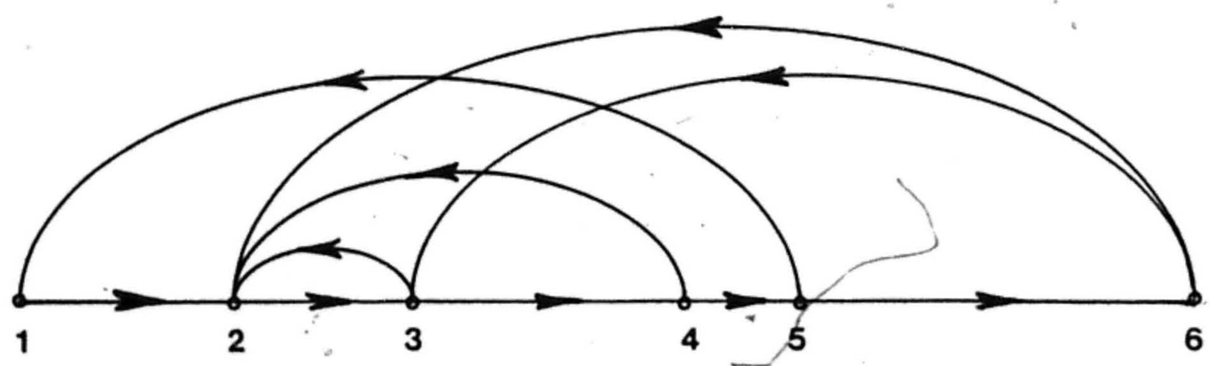

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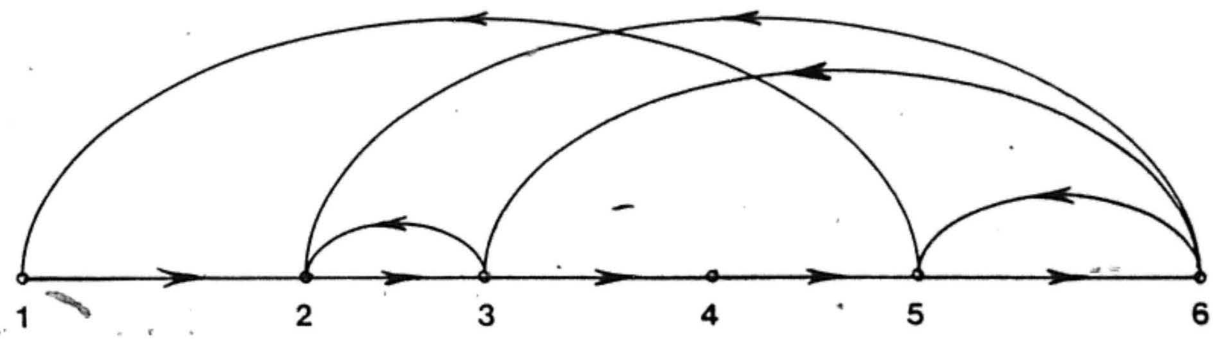

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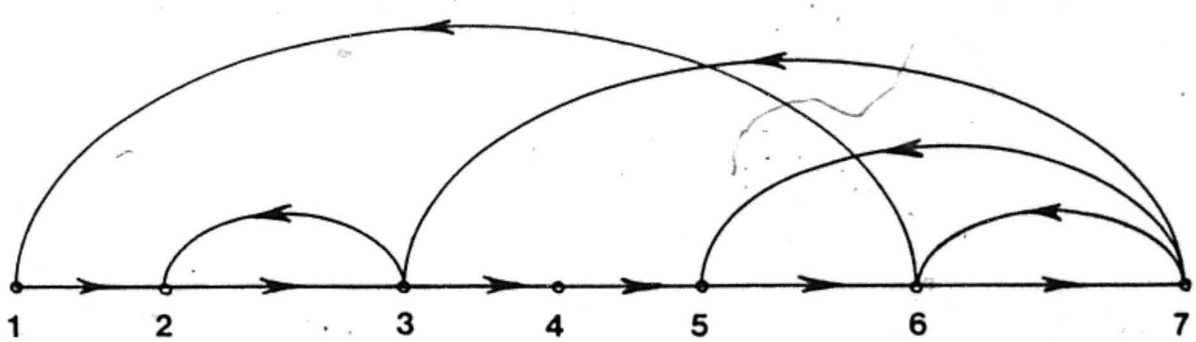

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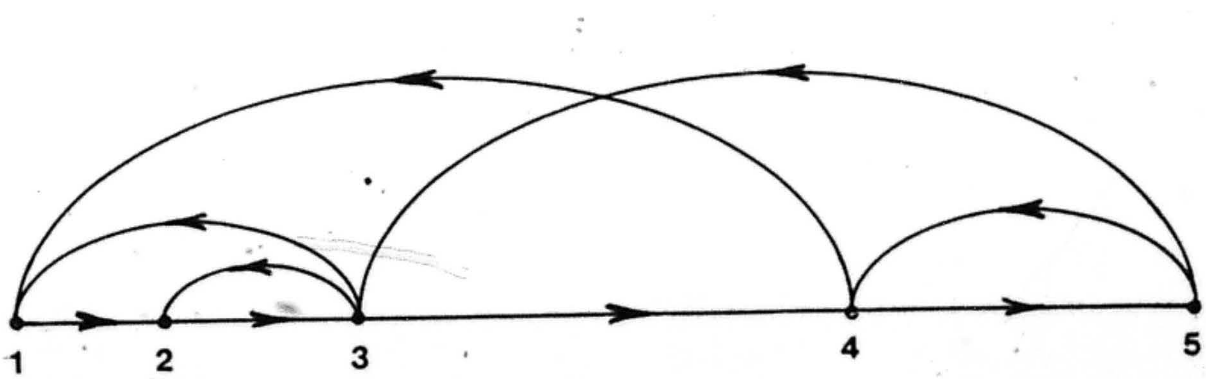

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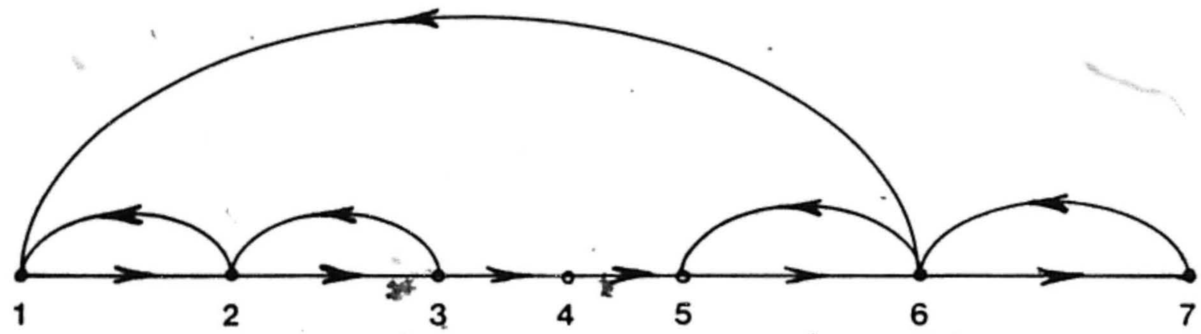

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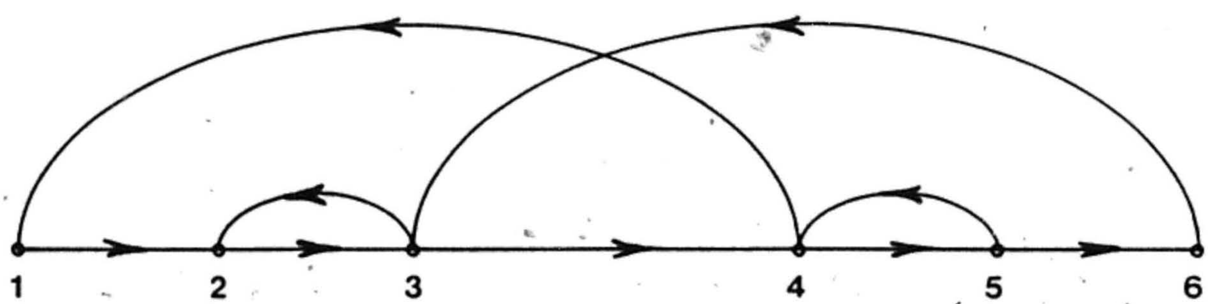

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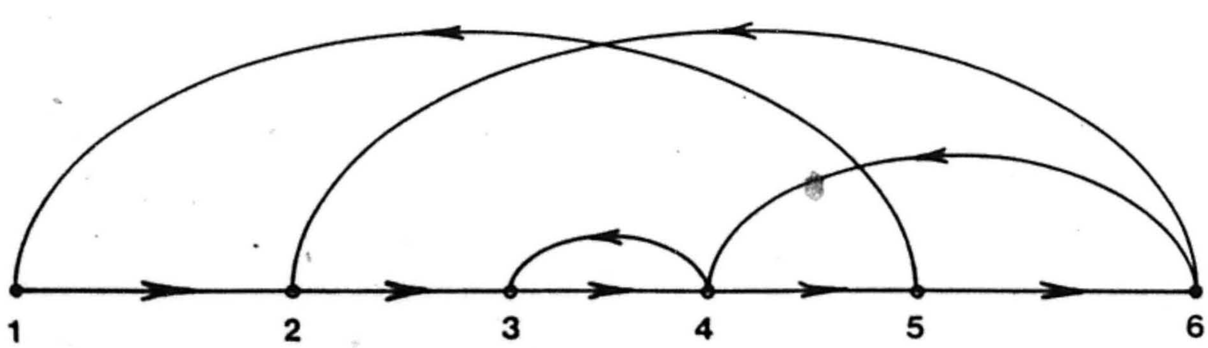

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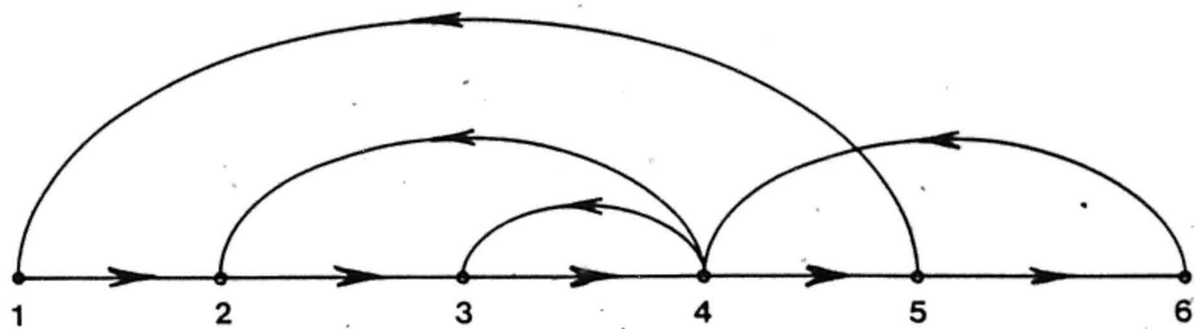

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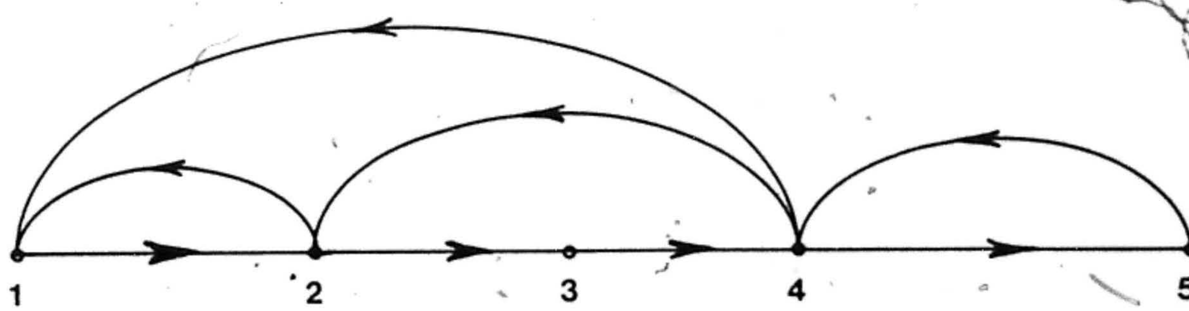

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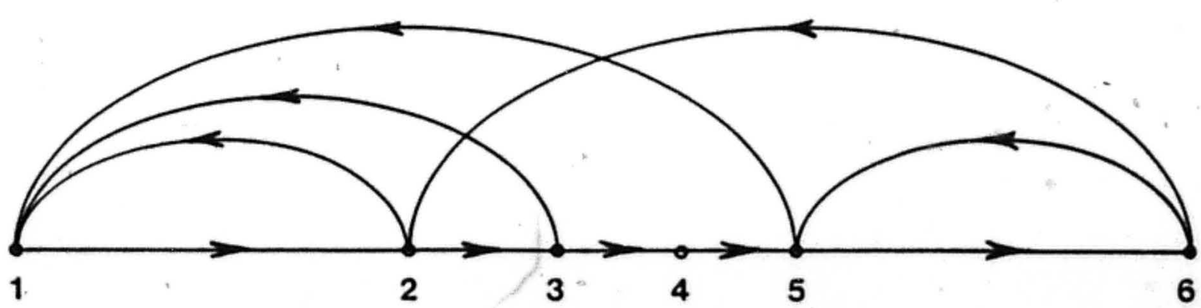

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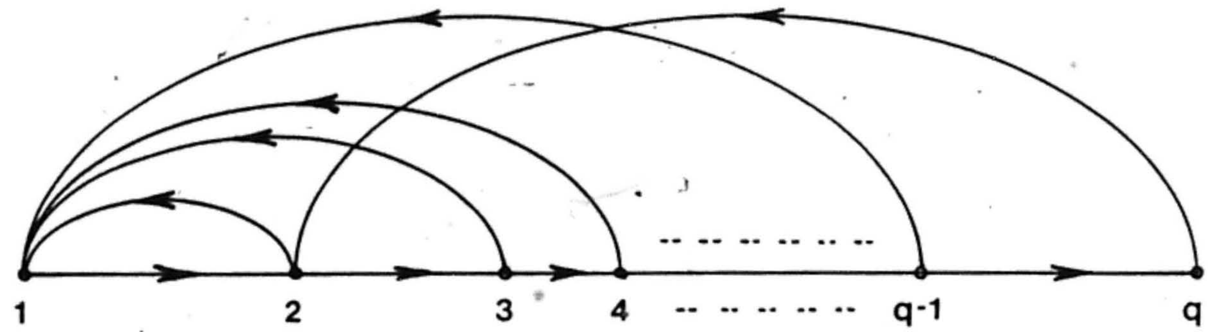

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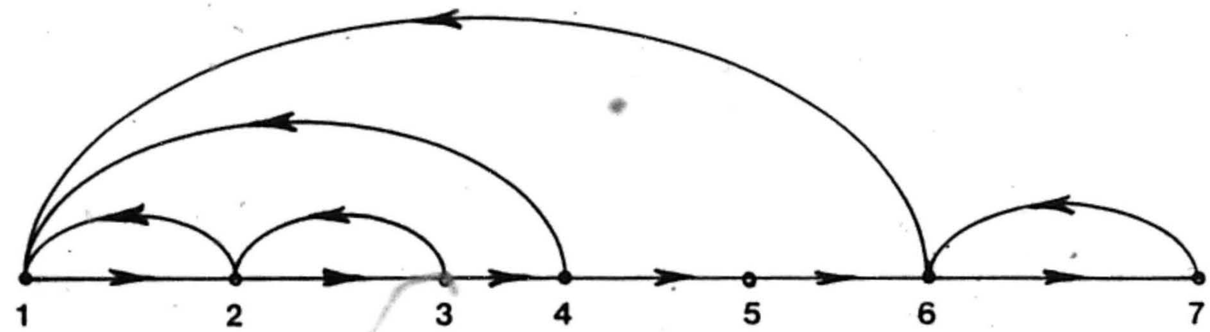

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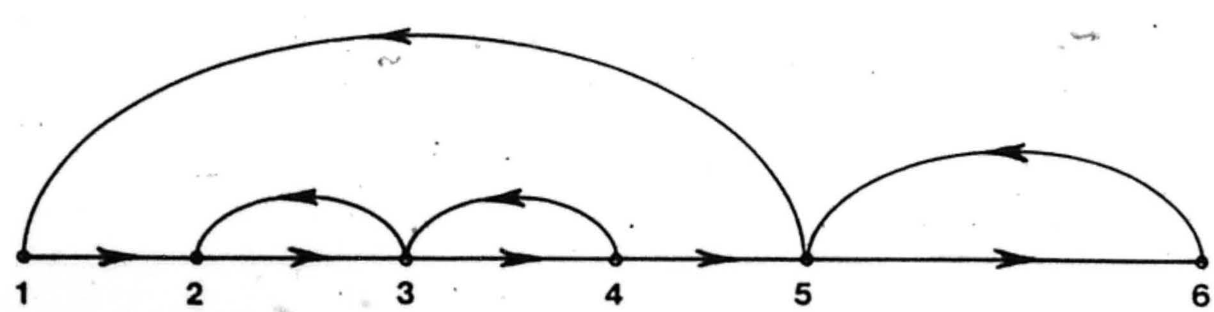

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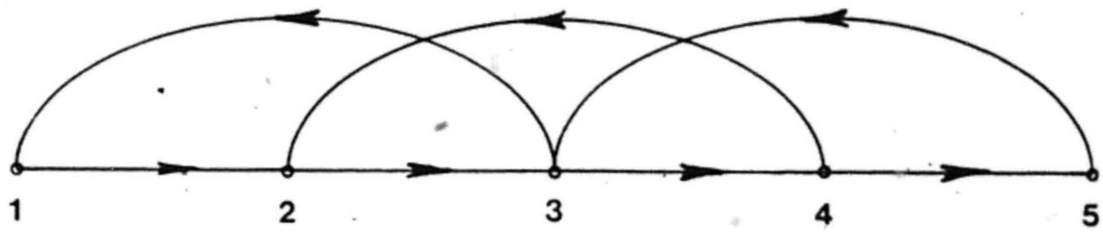

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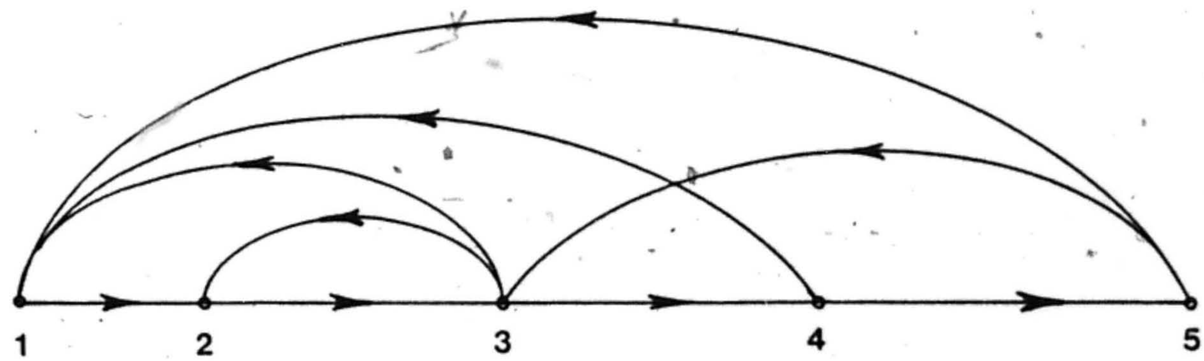

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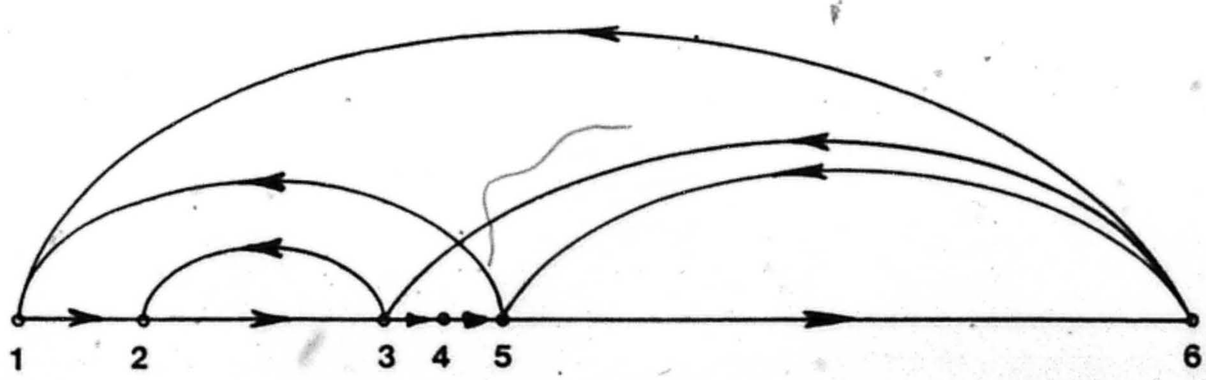

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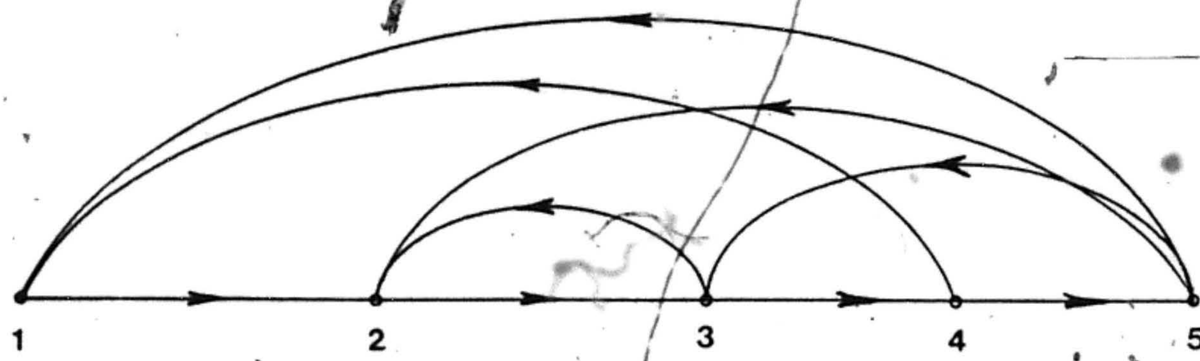

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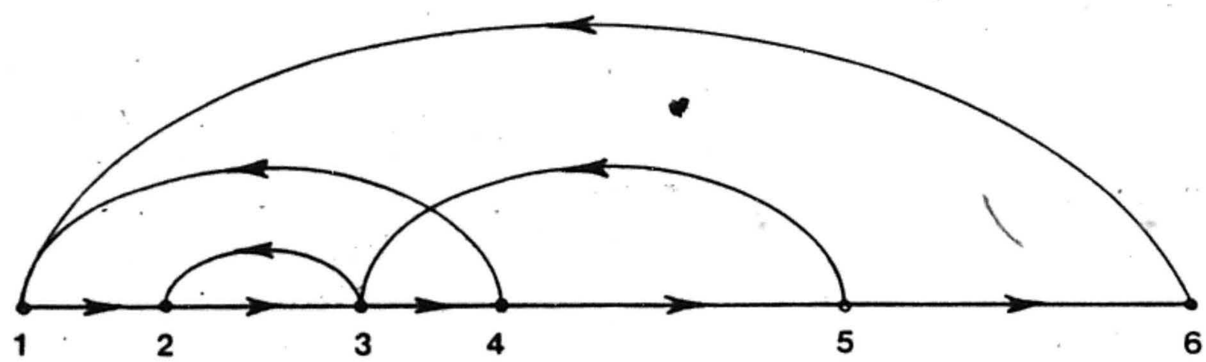

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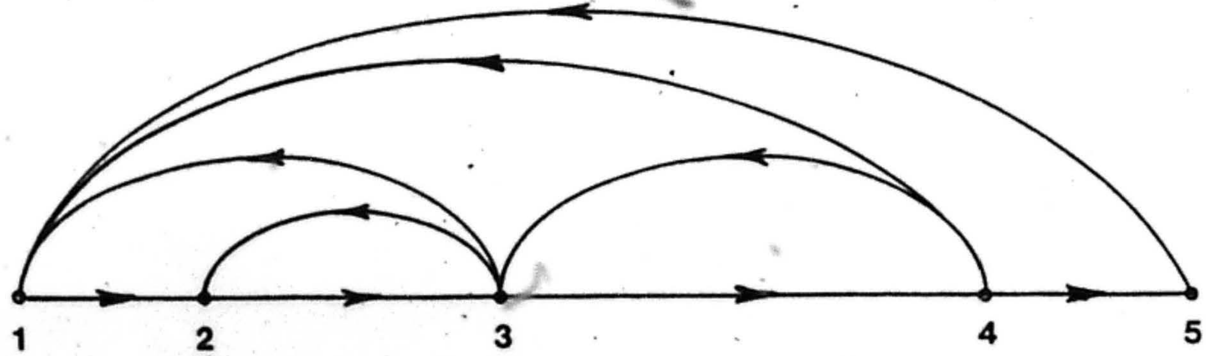

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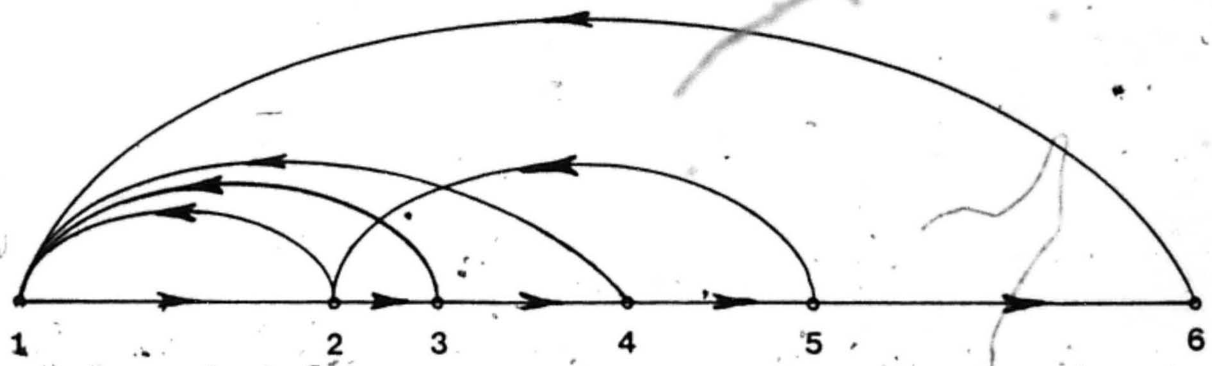

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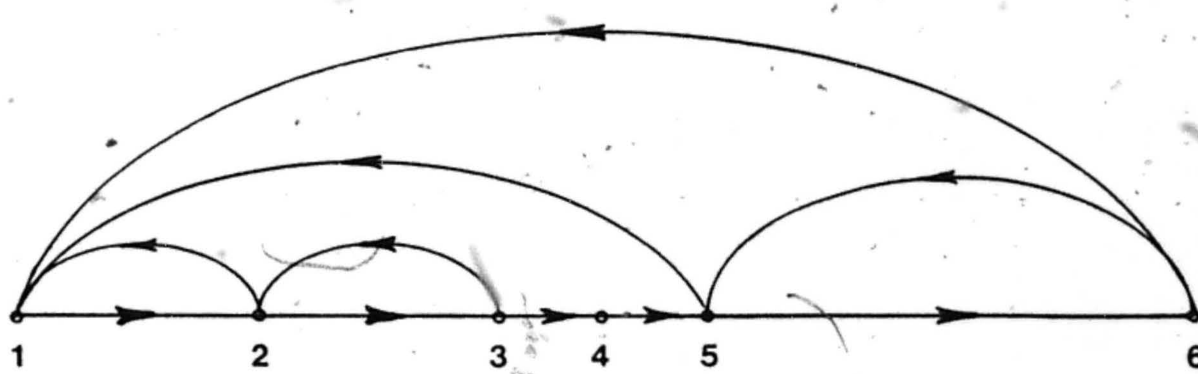

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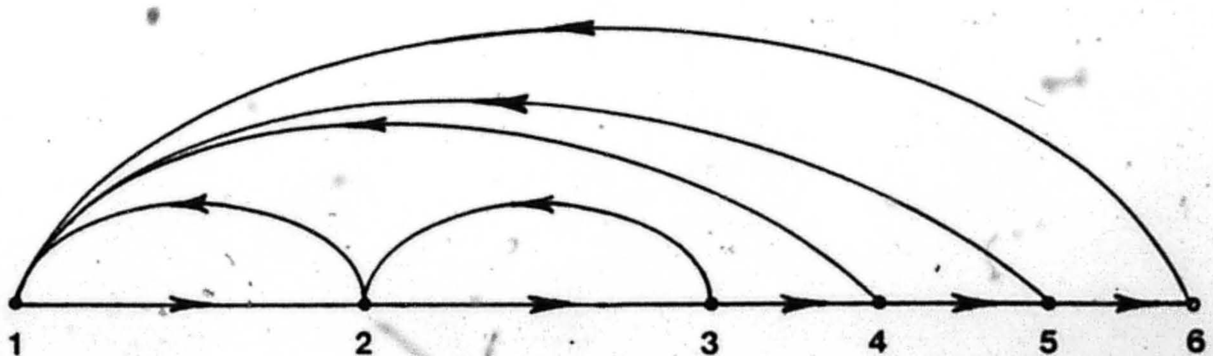

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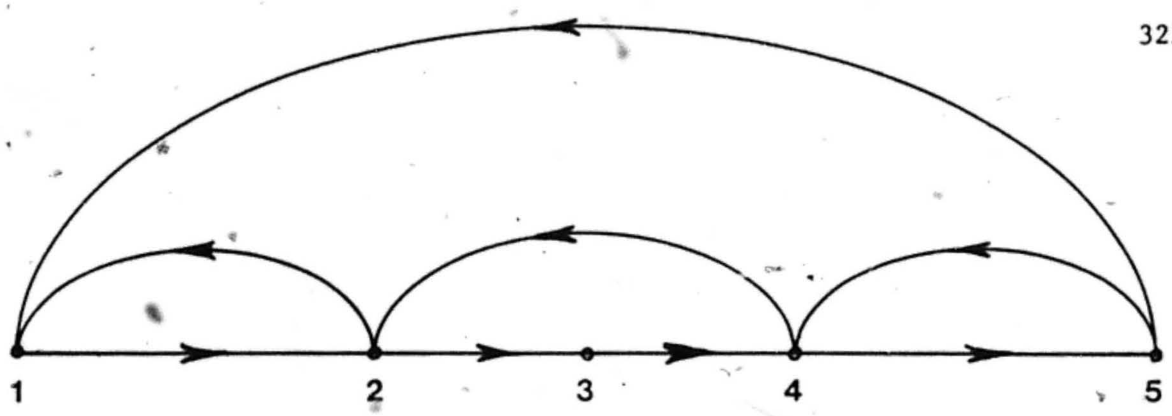

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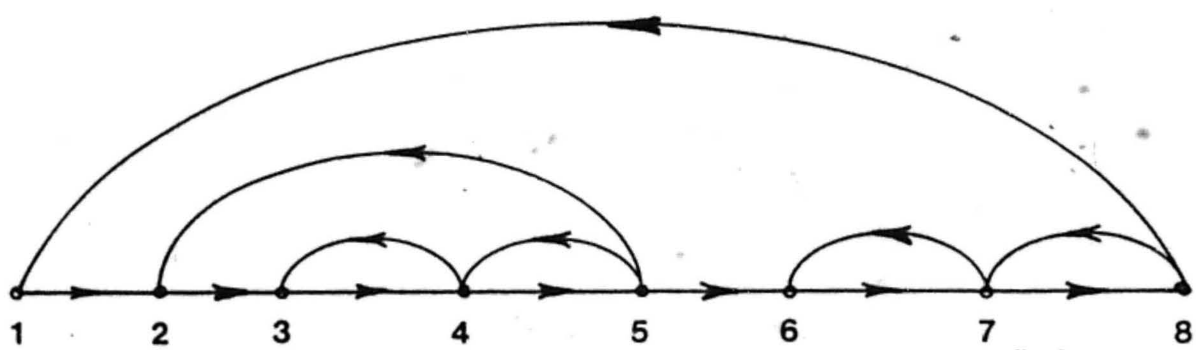

MAX.26

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