THE UNIVERSITY OF CALGARY

NON REPETITIVE WALKS IN GRAPHS AND DIGRAPHS

by

JAMES D. CURRIE

A THESIS
SUBMITTED TO THE FACULTY OF GRADUATE STUDIES
IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
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DEPARTMENT OF MATHEMATICS AND STATISTICS

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FACULTY OF GRADUATE STUDIES

The undersigned certify that they have read, and recommend to
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Abstract

A word $w$ over alphabet $\Sigma$ is **non-repetitive** if we cannot write $w = abbc; a, b, c \in \Sigma^*$, $b \neq a$. That is, no subword of $w$ appears twice in a row in $w$. In 1906, Axel Thue, the Norwegian number theorist, showed that arbitrarily long non-repetitive words exist on a three letter alphabet.

Call graph or digraph $G$ **versatile** if arbitrarily long non-repetitive words can be walked on $G$. This work deals with two questions:

1. Which graphs are versatile?
2. Which digraphs are versatile?

Our results concerning versatility of digraphs may be considered to give information about the structure of non-repetitive words on finite alphabets.

We attack these questions as follows:

(I) We introduce a partial ordering of digraphs called **mimicking**. We show that if digraph $G$ mimics digraph $H$, then if $H$ is versatile, so is $G$.

(II) We then produce two sets of digraphs MIN and MAX, and show that every digraph of MIN is versatile (These digraphs are intended to be minimal in the mimicking partial order with respect to being versatile.) and no digraph of MAX is versatile. (The digraphs of MAX are
intended to be maximal with respect to not being versatile.

(III) In a lengthy classification, we show that every digraph either mimics a digraph of MIN, and hence is versatile, or "reduces" to some digraph mimicked by a digraph of MAX, and hence is not versatile.

We conclude that a digraph is versatile exactly when it mimics one of the digraphs in the finite set MIN. The set MIN contains eighty-nine (89) digraphs, and the set MAX contains twenty-five (25) individual digraphs, and one infinite family of digraphs.
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SOLO DEO GLORIA
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Chapter 1: Introduction

Definitions and Preliminaries:

A word is a finite sequence of elements of some finite set $\mathcal{I}$. We call the set $\mathcal{I}$ an alphabet, the elements of $\mathcal{I}$ letters. The set of all words over $\mathcal{I}$ is denoted by $\mathcal{I}^*$, the set of words of positive length over $\mathcal{I}$ by $\mathcal{I}^+$. We take a naive view of words as strings of letters; thus the concatenation of two words $w$ and $v$, written $wv$, is simply the string consisting of the letters of $w$ followed by the letters of $v$. Say that $v$ is a subword of $w$ if we can write $w = uvz$; $u, v, z \in \mathcal{I}^*$. We say $v$ is a prefix (suffix) of $w$ if we can write $w = vz (zv)$; $v, z \in \mathcal{I}^*$. The empty word, denoted $\epsilon$, is the word with no letters in it. Denote by $|w|$ the length of $w$, equal to the number of letters in $w$.

Let $\mathcal{I}, \mathcal{J}$ be alphabets. A substitution $h: \mathcal{I}^* \rightarrow \mathcal{J}^*$ is a function generated by its values on $\mathcal{I}$. That is, if $w$ is a word on $\mathcal{I}$, $w = a_1a_2 \ldots a_m$, $a_i \in \mathcal{I}$, $1 \leq i \leq m$ then $h(w) = h(a_1)h(a_2) \ldots h(a_m)$.

Define a word of type $\omega$, to be a countable sequence of letters over some alphabet $\mathcal{I}$. If $h: \mathcal{I}^* \rightarrow \mathcal{I}^*$ is some substitution with $b$ a prefix of $h(b)$ for some $b \in \mathcal{I}$, and $h(b)$ longer than $b$, then denote by $h^\omega(b)$ be the word of
type \( w \) having initial segment \( h^n(b) \) for every \( n \). This limit makes sense, as \( h^n(b) \) will be a prefix of \( h^{n+1}(b) \) for each \( n \).

A word \( w \) over alphabet \( Z \) is non-repetitive if we cannot write \( w = abbc; a, b, c \in Z^*, b \neq c \). That is, no subword of \( w \) appears twice in a row in \( w \). The term square-free is also used for such words in the literature.

For the purposes of this thesis, a graph (digraph) \( G = (V, E) \) consists of a finite set \( V \) of vertices, together with a set \( E \) of unordered (ordered) pairs of vertices. If \( G \) is a graph, denote by \( \text{vert}(G) \) the set of vertices of \( G \). If \( a, b \in V \), and \((a, b) \in E\), then we say the edge \( ab \) is in \( G \). An edge of the form \( aa \), \( a \in V \), is called a loop. For technical reasons to become apparent later (See Lemma 3.5), we allow digraphs to contain loops. However, we only consider graphs not containing loops.

If \( G \) is a graph or digraph we may consider \( V = \text{vert}(G) \) to be an alphabet. We say that the word \( w \in V^* \) is a walk on \( G \) if whenever \( ab \) is a two letter subword of \( w \), then \( ab \) is an edge of \( G \). We say that \( w \) can be walked on \( G \), or \( G \) allows walk \( w \). A graph or digraph \( G \) is called versatile if arbitrarily long non-repetitive
words can be walked on \( G \). This work deals with two questions:

(1) Which graphs are versatile?
(2) Which digraphs are versatile?

**Background:** In 1906, Axel Thue, the Norwegian number theorist, showed that arbitrarily long non-repetitive words exist on a three letter alphabet. (See [19].) This result has been rediscovered many times, by Arshon [1], Morse and Hedlund [12] and Hawkins and Mientka [10], for example.

This result of Thue is counter-intuitive, and interesting for its own sake. It is also useful for the construction of pathological objects and counterexamples. An important example of a use of Thue’s result is in the solution of the Burnside problem by Novikov and Adjan [13].

There is a large literature concerning non-repetitive words (See the bibliography of Bean, Ehrenfeucht and McNulty [3].) By König’s lemma, the existence of arbitrarily long non-repetitive words on a finite alphabet is equivalent to the existence of a non-repetitive word of type \( \omega \) on that alphabet. Shelton
and Soni [16], [17], [18] investigate the structure of
the set of non-repetitive \( \omega \) words on a three letter
alphabet, showing the set to be perfect with respect to a
natural metric.

Call a word \( \omega \) over alphabet \( \Sigma \) \textit{strongly cube-free} if
we cannot write \( \omega = ab\beta c \), where \( a, b, c \in \Sigma^* \), \( \beta \in \Sigma \), and
\( \beta \) is the first letter of \( b \). If \( \Sigma \) is a two letter alphabet
and \( \tau \) a three letter alphabet, then a strongly cube-free
word of type \( \omega \) over \( \Sigma \) gives rise to a non-repetitive word
of type \( \omega \) over \( \tau \) in a natural way, and vice versa. (See
Braunholtz [5].) Fife [9] shows that the strongly
cube-free words of type \( \omega \) over a two letter alphabet form
a Cantor set under a natural metric.

The study of words which are non-repetitive or
strongly cube-free is generalized in Bean, Ehrenfeucht
and McNulty [3]. Here the question of words avoiding an
arbitrary pattern is considered. A word \( \omega \in \Sigma^* \) \textit{avoids} the
word \( \nu = b_1 b_2 \ldots b_m \) if we cannot write \( \omega = ah(b_1 b_2 \ldots b_m)c \)
where \( a, c \in \Sigma^* \), and \( h \) is a substitution not mapping any
of the \( b_i \) to the empty word. An algorithm is given to
determine whether, given \( \nu \), there exists a natural number
\( n \), so that there exist arbitrarily long words avoiding \( \nu \)
on an \( n \) letter alphabet. If such an \( n \) exists, \( \nu \) is said
to be \textit{avoidable}. If \( \nu \) is avoidable, it is natural to
attempt to bound the $n$ mentioned above. This problem is attacked in the paper Baker, McNulty, Taylor [2]. From [2], the following question naturally arises: On which directed graphs can arbitrarily long non-repetitive words be walked?

As mentioned, this question is the subject of the present thesis. In a different light, one may consider this question to be in the spirit of the investigations of Shelton, Soni and Fife: What can we say about the structure of non-repetitive words?

Let $w$ be a word of type $\omega$ over alphabet $\Sigma$. Baker, McNulty and Taylor define the transition digraph of $w$ to be that digraph having vertex set $\Sigma$, and an edge $a_i a_j$, $a_i, a_j \in \Sigma$, exactly when $a_i a_j$ is a subword of $w$. It is shown in [2] that if $w$ is a non-repetitive word of type $\omega$ on the three letter alphabet $\{a, b, c\}$, then $w$ must have a transition digraph with edges $ab, ba, ac, ca, bc, cb$. Equivalently, a digraph on vertices $a, b, c$ is versatile only if it contains the six edges $ab, ac, ba, bc, ca, cb$. Our results concerning versatility of digraphs may thus be considered to give information about the structure of non-repetitive words on finite alphabets.

Choffrut and Culik [7] consider the following
problem: Let $Z$ be a finite alphabet, $w_1, w_2, \ldots, w_m$ words over $Z$. Do there exist arbitrarily long words over $Z$ not including any of the $w_i$ as subwords? Thus while Bean, Ehrenfeucht and McNulty consider the problem of avoiding patterns, Choffrut and Culik wish to avoid specific words. The present work may be considered a hybrid of these two approaches: If $D$ is a digraph with vertices $v_1, v_2, \ldots, v_n$, we wish if possible to find arbitrarily long words on $\{v_1, v_2, \ldots, v_n\}$ avoiding the pattern $xx$, and simultaneously avoiding the specific words $v_i v_j$, where $v_i v_j$ is any non-edge of $D$.

Outline: Having motivated our work in the previous section, we make some remarks concerning our attack:

(I) We introduce a partial ordering of digraphs called mimicking. We show that if digraph $G$ mimics digraph $H$, then if $H$ is versatile, so is $G$.

(II) We then produce two sets of digraphs MIN and MAX, and show that every digraph of MIN is versatile (These digraphs are intended to be minimal in the mimicking partial order with respect to being versatile.) and every digraph of MAX is not versatile. (The digraphs of MAX are intended to be maximal with respect to not being versatile.)

(III) In a lengthy classification, we show that
every digraph either mimics a digraph of MIN, and hence is versatile, or "reduces" to some digraph mimicked by a digraph of MAX, and hence is not versatile.

We conclude that a digraph is versatile exactly when it mimics one of the digraphs in the finite set MIN.

Step (II) naturally presupposes the construction of certain non-repetitive words. From Axel Thue on down, those wishing to construct squarefree words have used substitutions. A substitution $h : X^* \rightarrow X^*$ is called square-free if whenever $w \in X^*$ is non-repetitive, so is $h(w)$. Axel Thue showed that the substitution $h : \{a, b, c\}^* \rightarrow \{a, b, c\}^*$ given by

\[ h(a) = abcab \]
\[ h(b) = acabcb \]
\[ h(c) = acbcacb \]

is squarefree. It follows that $h^\omega(a)$ is a non-repetitive word of type $\omega$ on $X$. On the other hand, the substitution $g : \{a, b, c\}^* \rightarrow \{a, b, c\}^*$ given by

\[ g(a) = c \]
\[ g(b) = bca \]
\[ g(c) = ba \]

is not square-free. In fact $g(bcb) = bcababca$, which contains the repetition $abab$. Nonetheless, the fact that $g^\omega(b)$ is non-repetitive was proved by Arshon [1] in the
1930's. Crochemore [8], defines a concept of weak square-freeness for substitutions. Let $\Sigma$ be an alphabet. Then $h: \Sigma^* \rightarrow \Sigma^*$ is weakly square-free if there exist $x, w$, where $x \in \Sigma, w \in \Sigma^+$, such that $h(x) = xw$, and $h^\omega(x)$ is non-repetitive. Although $g$ is not square-free, $g$ is weakly square-free.

Let $f: \{1, 2, 3\}^* \rightarrow \Sigma^*$ be a substitution. In the body of this thesis (see Lemma 2.4.), we prove that under certain conditions on $f$, $f(g^\omega(b))$ is non-repetitive, with $g$ given as above. These conditions do not force $f$ to be square-free, in fact $f(bcb), f(aca)$ are explicitly allowed to contain repetitions. This result is used to produce non-repetitive words of type $\omega$. Except in one case, all of the many non-repetitive walks used in this thesis are of the form $f(g^\omega(a))$ for such an $f$. In the other case we generate a non-repetitive word using weakly squarefree substitutions on a five element alphabet.

Much work has been done on square-free substitutions, and cube-free substitutions, which are defined analogously. References may be found in the bibliographies of Berstel [4] and Crochemore [8]. We give an example of a particularly beautiful result of Karhumaki [11]:

Theorem: Let $h : \{a, b\}^* \to \{a, b\}^*$ be a substitution such that $h(a)$ begins with an a. Then the word $h^\omega(a)$ is cube-free if and only if the word $h^{10}(a)$ is cube-free.

One last remark is in order, of interest to those following the work of Robertson, Seymour [15]: One might ask why we consider graphs separately from digraphs, since a graph $G$ may be considered to be simply a symmetric digraph. It turns out that the solution of the graph case of our problem allows us to find a nice classification scheme for digraphs. Moreover, it follows from the work of Robertson, Seymour on graph minors that the graph case will have a nice solution: From the weaving lemma of chapter 2 one may deduce that if $G$ does not allow arbitrarily long non-repetitive walks, then neither does any minor of $G$. Thus [15] implies that there is an excluded minor characterization of those graphs not allowing arbitrarily long non-repetitive walks. We know of no generalization of the work of [15] to digraphs.

Open Problems: (1) It was remarked above that of the digraphs on three vertices $a$, $b$, $c$, only a digraph including edges $ab$, $ba$, $bc$, $cb$, $ca$, $ac$ allows arbitrarily long non-repetitive walks. We can show that if $w$ is a
A non-repetitive word of type \( \omega \) on three letters \( a, b, c \), then \( \omega \) must contain as subwords all of the words in one of the following sets (up to a permutation of letters):

\[
\begin{align*}
H_1 &= \{ \text{aba, abc, acb, bab, bac, bca, cac, cba, cbc} \} \\
H_2 &= \{ \text{abc, aca, acb, bac, bca, bcb, cac, cab, cba, cbc} \}
\end{align*}
\]

A non-repetitive word of type \( \omega \) all of whose three letter subwords are in \( H_1 \) is \( g^\omega(\omega) \) where \( g \) is Arshon's substitution, given above.

A non-repetitive word of type \( \omega \) all of whose three letter subwords are in \( H_2 \) is \( g(f^\omega(1)) \) where \( f, g \) are given by

\[
\begin{align*}
f(1) &= 142 \\
f(2) &= 1435 \\
f(3) &= 143532 \\
f(4) &= 1532 \\
f(5) &= 1535 \\
g(1) &= ac \\
g(2) &= acb \\
g(3) &= acbc \\
g(4) &= abc \\
g(5) &= abcb
\end{align*}
\]

That \( g(f^\omega(1)) \) is non-repetitive may be proved using
the methods of Chapter 7 although this fact is not used in this thesis. In general, if \( w \) is a non-repetitive word of type \( \omega \) on \( n \) letters, what \( k \)-letter subwords must \( w \) contain? (This question could be phrased in the language of hypergraphs.)

(2) Call a word \( w \) strongly non-repetitive if we cannot write \( w = abcd, a, b, c, d \in \mathbb{Z}^+, b \neq c, \) a permutation of \( b \). There exists a strongly non-repetitive word of type \( \omega \) on a five letter alphabet. Whether such a word exists on four letters is an open problem. (See [8], [14].) On which digraphs can arbitrarily long strongly non-repetitive words be walked?
Chapter 2: Graphs

We start this chapter with some definitions concerning graphs and digraphs.

Let G be a graph (digraph) with vertex set V, a, b ∈ V. We say that the word p ∈ (V \ {a, b})* is a (directed) path in G from a to b if the word apb is a walk in G, and no vertex of G appears in p twice. The graph P_i whose vertex set is {1, 2, ..., i} and whose edges are 12, 23, ..., (i-1)i, is called the path on i vertices.

A graph or digraph G is connected if for every a, b ∈ V, a ≠ b, there is either a path in G from a to b, or a path in G from b to a. A digraph G is strongly connected if for every a, b ∈ V, there is a path in G from a to b and a path in G from b to a.

Let G be a graph (digraph) with vertex set V, a ∈ V. Let p ∈ (V \ {a})* be a word, p ≠ ε. If no vertex of V appears twice in p, and both ap and pa are walks in G, then we say that the word ap is a cycle of G based at a, or simply, a cycle of G. (Various terms exist in the literature. Others are circuit, and simple cycle.) A graph C whose vertices are {c_1, c_2, ..., c_m} and whose edges are c_1c_2, c_2c_3, ..., c_{m-1}c_m, c_mc_1 is called a cycle.
If $G$ is a graph (digraph), $a, b \in V$, then if $ab$ is an edge of $G$, say that $b$ is a neighbour of $a$ (b is a successor of $a$, $a$ is a predecessor of $b$). The degree (indegree, outdegree) of $a$ is the number of neighbours (predecessors, successors) of $a$ in $G$.

If $G_1, G_2$ are graphs (digraphs) with vertex sets $V_1, V_2$ and edge sets $E_1, E_2$ then denote by $G_1 \cap G_2$ the graph with vertex set $V_1 \cap V_2$ and edge set $E_1 \cap E_2$. Analogously define $G_1 \cup G_2$.

In this chapter, we answer the question: Which graphs are versatile? We restrict our attention to connected graphs, since a word $v$ can be walked on a graph $G$ if and only if $v$ can be walked on a connected component of $G$. We prove the following theorem:

**Theorem 2.1:** A connected graph $G$ is versatile unless $G$ is a path on four or fewer vertices.

The following observation proves useful.

**Lemma 2.1 (a) (Weaving Lemma):** Let $v = a_1a_2...a_r$ be a non-repetitive word, $a_1, a_2, ..., a_r \in S$, some alphabet. Let $b_1, b_2, ..., b_{r+1}$ be non-repetitive words on alphabet $T$, where $S$ and $T$ are disjoint. We permit some or all of the $b_i$ to be empty. Then $w = b_1a_1b_2a_2...b_rb_r b_{r+1}$ is a
non-repetitive word.

Proof: Suppose $w$ contains a repetition, say $w = uyzy$, $y \neq \epsilon$. Then $yy$ contains some $a_j$, for otherwise $yy$ is a subword of one of the $b_i$, contradicting the fact that the $b_i$ are non-repetitive.

Now if $p$ is a word on $S \cup T$, denote by $p|S$ the word formed by deleting from $p$ all the letters of $T$. Thus the above paragraph remarks that $y|S \neq \epsilon$; however, $a_1a_2...a_r = w|S = u|S^y|S^y|S^z|S$ and therefore $v = a_1a_2...a_r$ contains a repetition, namely $y|S^y|S$, which is a contradiction.

We thus conclude that $w$ is a non-repetitive word. \[ \square \]

Let $v$ be a word of type $\omega$ on some alphabet $S$, $S = \{ a_1, a_2, ..., a_n \}$. Let $G$ be a graph (digraph) including $S$ among its vertex set. Suppose that whenever $a_i a_j \in S^*$ is a subword of $v$ there is a path $P(a_i, a_j)$ in $G$ from $a_i$ to $a_j$ such that no vertex of $P(a_i, a_j)$ is in $S$. We say that $v$ can be walked in $G$ modulo paths. The weaving lemma will often be applied in the following way:

**Lemma 2.1 (b) (Second Weaving Lemma):** Let $v$ be a non-repetitive word of type $\omega$, $G$ a graph (digraph). If $v$ can be walked on $G$ modulo paths, then $G$ is versatile.

**Proof:** Pick $n > 0$. Let $b_1b_2...b_n$ be the initial segment of $v$ of length $n$. The word $w$ where
\[ w = b_1 P(b_1, b_2) b_2 P(b_2, b_3) b_3 \ldots b_{n-1} P(b_{n-1}, b_n) b_n \]

will be a non-repetitive word by the weaving lemma. By construction, \( w \) is a non-repetitive walk on \( G \) of length \( n \) or more. Thus \( G \) allows arbitrarily long non-repetitive walks. \( \Box \)

We now commence the proof of Theorem 2.1, proving a series of lemmas.

**Lemma 2.2:** Let \( G \) be a graph with a vertex \( v \) with degree \( (v) \geq 3 \). Then \( G \) is versatile.

**Proof:** Let three neighbors of \( v \) be \( a, b, c \). Let \( w \) be any non-repetitive word of type \( w \) on \( \{ a, b, c \} \). Then \( w \) can be walked on \( G \) modulo paths, with

\[
P(a, b) = P(b, a) = P(b, c) = P(c, b) = P(c, a) = P(a, c) = v
\]

(See Figure 2.1) Thus, by the second weaving lemma, \( G \) is versatile. \( \Box \)

Restating Lemma 2.2, any graph which is not versatile must have the degree of every vertex being 2 or less. In the case of connected graphs, we are left with paths and cycles.

**Lemma 2.3:** Let \( C = c_1 c_2 \ldots c_m \) (\( m \geq 3 \)) be a cycle. Then \( C \) is versatile.

**Proof:** Again we use the second weaving lemma. Here let \( v \) be any non-repetitive word of type \( w \) on \( \{ c_1, c_2, c_3 \} \). Then \( v \) can be walked on \( C \) modulo paths,
Figure 2.1
where

\[ P(c_1c_2) = P(c_2c_1) = P(c_2c_3) = \varepsilon P(c_3c_2) = \varepsilon \]
\[ P(c_3c_1) = c_4c_5\ldots c_m \]
\[ P(c_1c_3) = c_m c_{m-1}\ldots c_4 \]

(See Figure 2.2) Thus \( G \) is versatile.\( \diamond \)

We have seen that every connected graph which is not a path is versatile. To conclude our examination of graphs we consider paths. Paths on four or fewer vertices do not allow arbitrarily long non-repetitive walks. It suffices to show this for \( P_4 \), since \( P_4 \) contains shorter paths as subgraphs.

Suppose that \( P_4 \) allows arbitrarily long non-repetitive walks. Then let \( v \) be a non-repetitive word of type \( \omega \) which can be walked on \( P_4 \). We chop \( v \) up into blocks starting with 1. That is, consider the possible subwords of \( v \) commencing with 1, ending with 2, and containing exactly one 1. (See Figure 2.3). Clearly these are \( a = 12, b = 1232, c = 123432 \). However a moment's thought shows that block \( a \) cannot appear in \( v \) since the words \( aa, ab, ac \) all contain the repetition \( aa \), and if \( v \) contains block \( a \), then it must contain one of these longer words.

Thus \( v \) must be composed entirely of the two blocks \( b \) and \( c \). However any non-repetitive word on two letters is
Figure 2.3
finite, hence \( v \) must be a finite word. This is a contradiction. Thus \( P_4 \) does not allow arbitrarily long non-repetitive walks.

**Definition:** Let \( S = \{ x_1, x_2, x_3 \} \), \( T \) be alphabets and let \( h: S^* \rightarrow T^* \) be a substitution. Say that \( h \) is *suitable* if

1) \(| h(x_i) | \leq | h(x_j) | + | h(x_k) | \) for \( 1 \leq i, j, k \leq 3 \), \( i, j, k \) distinct.

2) For \( 1 \leq i \leq 3 \) one cannot write \( h(x_i) = uw = wz \), \( u, w, z \in T^* \), \( u, w, z \neq \epsilon \).

3) If \( w \in S^* \) is a non-repetitive word with \( | w | = 3 \) and \( w = x_2 x_3 x_2, x_1 x_3 x_1 \), then \( h(w) \) is non-repetitive.

To show that \( P_5 \) allows arbitrarily long non-repetitive walks, we introduce another lemma for producing new non-repetitive words from old. In fact this lemma will be one of the main tools of this thesis.

**Lemma 2.4 (Substitution Lemma):** Let \( S \) be the alphabet \( \{ x_1, x_2, x_3 \} \). Let \( v \in S^* \) be a non-repetitive word, such that \( x_2 x_3 x_2, x_1 x_3 x_1 \) are not subwords of \( v \). If \( h \) is suitable, then \( h(v) \) is non-repetitive.

**Proof:** Suppose \( v \) fulfills the conditions of the lemma and \( h \) is suitable. Let \( v = a_1 a_2 \ldots a_m \). For each \( i, 1 \leq i \leq m \), say \( h(a_i) = e_i \). For the sake of a contradiction,
suppose \( h(v) = e_1 e_2 \ldots e_m = abbc \), some \( a, b, c \in \mathbb{T} \), \( b \neq c \).

Without loss of generality, shortening \( v \) if necessary, write

\[
e_1 e_2 \ldots e_{j-1} e'_j = e'_j e_{j+1} \ldots e_{m-1} e'_m = b \ldots \text{(*)}
\]

where \( e_1 = e_1' e_1'' \), \( e_j = e_j' e_j'' \), \( e_m = e_m' e_m'' \)

\( e_1', e_j', e_m' \neq \epsilon \).

Since \( h(v) \) is repetitive, \( m > 3 \). Otherwise, by condition 3) of the definition of suitability, \( e_1 e_2 e_3 \) is \( x_1 x_3 x_1 \) or \( x_2 x_3 x_2 \), contrary to our assumptions on \( v \). Also \( j > 1 \), otherwise

\[
| e_1 | > | e_2 | + | e_3 | \text{ by line (*) and the fact that } m > 3 . \text{ Similarly, } j < m .
\]

Claim: The two expressions \( e_1 e_2 \ldots e_j \) and \( e'_j e_{j+1} \ldots e_m \) "match up" in the natural way, i.e.

\[
e'_j = e''_m , \quad e_j' = e''_j , \quad m = 2j - 1 \text{ and } \]

\[
e_1 + i = e_{j+i} \text{ for } i = 1 \text{ to } j-2 .
\]

Proof of Claim: If \( e_j' \neq e''_m \) suppose that

\[
| e'_m | > | e'_j | . \text{ Say that } e'_m = e''_k e_{k+1} \ldots e'_j \text{ for some } k < j , \text{ and } e_k = e''_k , e''_m \neq \epsilon . \text{ Then } h(a_k a_m) = e_k e_m = e'_j e''_k e''_{k+1} e_{k+2} \ldots e'_j e''_m \text{ which contains the repetition } e''_k e''_{k+1}. \]

By condition 3) on \( h \) we must have \( a_k = a_m \). However now
\[ e_k \cdot e_k = e_k \cdot e_m = e_m \cdot e_m = e_{k+1} \cdot e_m \cdot e_m \]

Note that \( e_k \neq e \), so that condition 2) is contradicted for \( h(a_k) \), which commences and ends with \( e_k \).

We get a similar contradiction if \( |e_m| < |e_j| \).

Thus \( e_m = e_j \) and \( e_{1} \ldots e_{j-1} = e_{j} \ldots e_{m-1} \). Repeating this argument we show that

\[ e_{1+i} = e_{j+i} \quad \text{for } i = 1 \ldots j-2. \]

\[ e_{j} = e_{1}', \quad \text{and } 2j - 1 = m, \quad \text{as desired.} \]

Note that \( e_{1+i} = e_{j+i} \) implies that \( a_{1+i} = a_{j+i} \), since \( h \) is suitable. From the claim,

\[ h(a_{1}a_{j}a_{m}) = e_{1}e_{j}e_{m} = e_{1}e_{j}e_{j}e_{j}e_{m}e_{m} \]

which repeats \( e_{1}e_{j} \). Since \( |a_{1}a_{j}a_{m}| = 3 \), one of the following cases must arise:

A: \( a_{1} = a_{j} \)

B: \( a_{j} = a_{m} \)

C: \( a_{1} = x_{1}, \quad a_{j} = x_{3}, \quad a_{m} = x_{1} \)

D: \( a_{1} = x_{2}, \quad a_{j} = x_{3}, \quad a_{m} = x_{2} \).

In case A, \( v \) contains the subword

\( a_{1}a_{2} \ldots a_{j-1}a_{1}a_{2} \ldots a_{j-1} \), which is a contradiction, as \( v \) is non-repetitive. Similarly case B cannot occur, as \( v \) would contain a repetition.

Suppose case C occurs. (Case D is similar.) Since
m \geq 4, \text{ and } m \text{ is odd, } m \geq 5. \text{ Therefore } j \geq 3.

Now \( a_2 = a_{j+1} \). But since \( v \) is non-repetitive,

\( a_2 \neq a_1 = x_1 \) and \( a_{j+1} \neq a_j = x_3 \). Thus \( a_2 = a_{j+1} = x_2 \).

Also \( a_{j-1} = a_{m-1} \) so that \( x_3 = a_j \neq a_{j-1} \) and

\( a_{m-1} \neq a_m = x_1 \). We conclude that \( a_{j-1} = a_{m-1} = x_2 \).

Therefore \( a_{j-1} a_{j+1} = x_2 x_3 x_2 \), contradicting our assumptions on \( v \).

The assumption that \( h(v) \) repeats leads to a contradiction. Therefore \( h(v) \) contains no repetition. \( \Box \)

**Remarks:** Several variations have been proved of a lemma with stronger conditions than the above, and a stronger conclusion. For example, in Bean, Ehrenfeucht and McNulty [3], the following lemma is proved.

**Lemma 2.5:** Let \( \Sigma, \Gamma \) be alphabets. Suppose that \( h: \Sigma^* \rightarrow \Gamma^* \) is a substitution such that

1') If \( x, y \in \Sigma \) and \( h(y) \) is a subword of \( h(x) \), then \( y = x \).

3') If \( w \in \Sigma^* \) is a non-repetitive word with \( |w| = 3 \) then \( h(w) \) is non-repetitive.

Then if \( v \in \Sigma^* \) is non-repetitive, so is \( h(v) \).

The proof is essentially that of Lemma 2.4, with
condition 1') sufficing to prove the claim. In fact our claim, with slight renaming, comes from [3]. We have stated this result of [3] as a lemma, as we will refer to it later.

Lemma 2.4, in comparison with Lemma 2.5, restricts h less, and v more. When \( \varepsilon = S \), condition 3') necessitates the checking of \( h(w) \) for twelve three letter words \( w \), whereas condition (3) only requires good behaviour from \( h \) on ten of these twelve triples.

Next we show how to produce arbitrarily long words \( v \) on \( S, S = \{ x_1, x_2, x_3 \} \) satisfying the conditions of the substitution lemma. Consider the substitution

\[
\begin{align*}
h &: S^* \rightarrow S^* \\
h(x_1) &= x_3 \\
h(x_2) &= x_2 x_3 x_1 \\
h(x_3) &= x_2 x_1
\end{align*}
\]

Clearly \( h \) meets conditions 1) and 2) of the definition of suitability. (We point out that \( h \) does not meet condition 1' above.) That \( h \) also meets condition 3) of suitability is verified by checking the action of \( h \) on triples of \( S \).

\[
\begin{align*}
h(x_1 x_2 x_1) &= x_3 x_2 x_3 x_1 x_3 \\
h(x_1 x_2 x_3) &= x_3 x_2 x_3 x_1 x_2 x_1
\end{align*}
\]
\[ h(x_1 x_3 x_1) = x_3 x_2 x_1 x_3 \]
\[ h(x_1 x_3 x_2) = x_3 x_2 x_1 x_2 x_3 x_1 \]
\[ h(x_2 x_1 x_2) = x_2 x_3 x_1 x_3 x_2 x_3 x_1 \]
\[ h(x_2 x_1 x_3) = x_2 x_3 x_1 x_3 x_2 x_1 \]
\[ h(x_2 x_3 x_1) = x_2 x_3 x_1 x_2 x_1 x_3 \]
\[ h(x_2 x_3 x_2) = x_2 x_3 x_1 x_2 x_1 x_2 x_3 x_1 \]
\[ h(x_3 x_1 x_2) = x_2 x_1 x_3 x_2 x_3 x_1 \]
\[ h(x_3 x_1 x_3) = x_2 x_1 x_3 x_2 x_1 \]
\[ h(x_3 x_2 x_1) = x_2 x_1 x_2 x_3 x_1 x_3 \]
\[ h(x_3 x_2 x_3) = x_2 x_1 x_2 x_3 x_1 x_2 x_1 \]

Only \( h(x_2 x_3 x_2) \) contains a repetition: \( x_1 x_2 x_1 x_2 \).

Let \( v \) be any non-repetitive word on \( S \). Any \( x_3 \) appearing internally in \( h(v) \) either comes from \( h(x_2) \) and appears in the context \( x_2 x_3 x_1 \), or comes from \( h(x_1) \) and appears in the context \( x_1 x_3 x_2 \). Thus the words \( x_1 x_3 x_1 \) and \( x_2 x_3 x_2 \) are not subwords of \( h(v) \).

Now suppose \( v \in S^* \) has no repetition and doesn't contain \( x_1 x_3 x_1 \) or \( x_2 x_3 x_2 \) as subwords. By the substitution lemma, \( h(v) \) contains no repetition. By our last observation, \( h(v) \) contains neither \( x_1 x_3 x_1 \) nor \( x_2 x_3 x_2 \). Thus by induction \( h^n(x_2) \) has no repetitions, and does not contain \( x_1 x_3 x_1 \) or \( x_2 x_3 x_2 \). We therefore see that the word \( h^n(x_2) \) fulfills the substitution lemma's conditions on \( v \), and can be made arbitrarily long.
We are now ready to show that $P_5$ allows arbitrarily long non-repetitive walks. Consider the following substitution.

$g: S^* \rightarrow T^*$

$g(x_1) = 12345432$

$g(x_2) = 123432345432123454323432$ (Sub 2.2)

$g(x_3) = 1234323454323432$

Clearly $g(v)$ is a walk on $P_5$ whenever $v \in S^*$. (See Figure 2.4)

Further, $g$ is suitable. The only condition difficult to check is condition 3). One must check these words for non-repetitiveness:

$g(x_1x_2x_1) = 1234543212343234543212345432343212345432$

$g(x_1x_2x_3) = 12345432123432345432123454323432123454323432$

$g(x_1x_3x_1) = 123454321234323454321234543234321234543212345432$

$g(x_1x_3x_2) = 12345432123432345432123454323432123454323432$
Figure 2.4
As an example, we show that $w = g(x_1 x_2 x_1)$ is non-repetitive. Suppose not. Then $w$ must contain a repetition $vv$. Being a repetition, $vv$ contains the symbol 1 exactly four, two or no times. We can rule out $vv$ containing no 1's, since then $vv$ would be entirely contained in one of $g(x_1)$, $g(x_2)$, $g(x_3)$, which can each be checked to be non-repetitive.

If $vv$ contains exactly four 1's, then the first and third 1's of $w$ are "matched" by $vv$:

$$\text{12345432123432345432123454323432}$$

However, as indicated in the above scheme, this cannot happen, as the subwords of $w$ commencing at the first and third 1's don't agree for long enough. (The extent of their agreement is underlined.)

Suppose $vv$ contains then exactly two 1's. If the
first 1 of \( w \) is contained in \( vv \), it must be matched with the second 1 of \( w \):

\[ 12345432 \overline{12343234543212345432343212345432} \]

We see that this is impossible.

Suppose that \( vv \) matches the second and third 1's of \( w \):

\[ 12345432 \overline{12343234543212345432343212345432} \]

Again we see that this is impossible; the underlined "zones" of agreement for these two 1's do not meet.

The second 1 of \( w \) cannot be matched with the fourth 1, since then \( vv \) would also contain the third 1. However, then \( vv \) would contain all four 1's, which is impossible, as mentioned.

The final possibility is that the third and fourth 1's of \( w \) should match. However, we note that \( w \) is a palindrome. Since the second and third 1's could not match, neither can the third and fourth.

By arguments of this type, all the listed words except for \( g(x_2x_3x_2) \) can be shown to be non-repetitive. Alternatively, \( g \) can be shown to be suitable by invoking the Long/Short Lemma of Chapter 7.

As \( g \) is suitable, \( g(h^n(x_2)) \) gives an arbitrarily
long non-repetitive walk on $P_5$ by choosing $n$ as large as
desired. Thus $P_5$ is versatile. Since any path on more
than five vertices contains $P_5$ as a subgraph, such paths
are also versatile. We have thus proved Theorem 2.5.

**Versatility of MIN.1 - MIN.4:** Since we have the
substitutions $h$ and $g$ handy, this is a convenient point
in the thesis at which to show that MIN.1 - MIN.4 are
versatile digraphs. Let $v = h^\omega(x_2)$, $w = g(h^\omega(x_2))$.

Recall from Chapter 1 the concept of a transition
digraph: given a word $u$ of type $\omega$ over a finite alphabet
$\Sigma$, the transition digraph of $u$ has as vertices those
letters of $\Sigma$ appearing in $u$, and a directed edge from
letter $x$ to letter $y$ exactly when $xy$ is a subword of $u$.
Thus MIN.1 is isomorphic to the transition digraph of $v$,
and MIN.3 is precisely the transition digraph of $w$. It
follows that MIN.1 and MIN.3 are versatile digraphs.

As we remarked earlier, $v$ does not contain subwords
$x_2x_3x_2$ or $x_1x_3x_1$. Whenever $x_3$ occurs in $v$ it is either in
the context $x_2x_3x_1$ or $x_1x_3x_2$. Let $v'$ be the word of type
$\omega$ arising from $v$ by replacing $x_3$ by $x_4$ whenever $x_3$ occurs
in context $x_1x_3x_2$. Clearly $v'$ will be a non-repetitive
word of type $\omega$. One checks that MIN.2 is isomorphic to the transition digraph of $v'$, and thus is versatile.

Similarly, one checks that $w$ does not contain subwords 232 or 434. Whenever 3 occurs in $w$ it is either in the context 234 or 432. Let $w'$ be the word of type $w$ arising from $w$ by replacing 3 by 3' whenever 3 occurs in context 432. Again $w'$ will be a non-repetitive word of type $\omega$. One checks that MIN.4 is the transition digraph of $w'$, and thus is versatile.
Chapter 3: Digraph Classification

In this chapter we ask the following question: Which digraphs are versatile? In analogy to chapter 2, we are only interested in strongly connected digraphs.

Lemma 3.1: Let $G$ be a digraph. Then $G$ is versatile if and only if one of $G$'s strongly connected components is versatile.

Proof: Clearly if a component of $G$ is versatile, so is $G$. Suppose that $G$ is versatile. Let $v$ be a non-repetitive word of type $w$ which can be walked on $G$.

We show that whenever $x$ and $y$ are vertices in different components of $G$ then a final segment of $v$ can be walked in one of $G \setminus \{x\}$ or $G \setminus \{y\}$. It will follow by induction on the size of $G$ that a non-repetitive walk of type $w$ exists in one of $G$'s components.

Suppose then that $x$ and $y$ are vertices of $G$ and there is no directed $xy$ path in $G$. If $v$ contains no $x$, then $v$ can be walked in $G \setminus \{x\}$ and we are done. If $v$ contains an $x$, then a final segment $v'$ of $v$ contains no $y$, and $v'$ can be walked in $G \setminus \{y\}$.

A strongly connected digraph can be written as a union of cycles. In the following lemma we relate the intersection of these cycles to the existence of
non-repetitive walks.

**Lemma 3.2 (Intersection Lemma):** Let $X$, $Y$ be directed cycles in the digraph $G$ so that $\text{vert}(X) \cap \text{vert}(Y) \neq \emptyset$. Then either

1) $X \cap Y$ is connected

or

2) $X \cup Y$ is versatile.

**Proof:** In fact if 1) does not hold, then $X \cup Y$ "contains" one of the versatile digraphs MIN 1 or MIN 2, in a sense to be made precise later. We show that 2).

First note that $X$ (similarly $Y$) gives a circular order to the vertices of $\text{vert}(X) \cap \text{vert}(Y)$.

**Case A:** The circular orders given to $\text{vert}(X) \cap \text{vert}(Y)$ by $X$ and $Y$ are different.

In this case there are vertices $x_1, x_2, x_3$ of $X \cap Y$ occurring in the order $x_1, x_2, x_3$ in the cycle $X$, and in the order $x_1, x_3, x_2$ in the cycle $Y$. Now we use the Second Weaving Lemma, Lemma 2.1(b). As in the last part of Chapter 2, let $v$ be $h^\omega(x_2)$. The Second Weaving Lemma requires us to walk $v$ on $X \cap Y$ modulo paths. We let the paths $P(x_1, x_2)$, $P(x_2, x_3)$, $P(x_3, x_1)$ be arcs in cycle $X$. We require that none of these paths contain $x_1$, $x_2$ or $x_3$. However, this is fulfilled because of the assumed circular order of these vertices in $X$. For example, the vertex $x_3$ cannot be on the arc of $X$ between
The required paths \( P(x_1, x_3), P(x_3, x_2), P(x_2, x_1) \) are chosen in \( Y \). Then \( v \) can be walked on \( X \cap Y \) modulo these paths, and by the Second Weaving Lemma, \( X \cap Y \) is versatile.

**Case B:** The circular orders on \( \text{vert}(X) \cap \text{vert}(Y) \) given by \( X \) and \( Y \) are the same. Suppose that \( X \cap Y \) is not connected. Then choose vertices \( x_1, x_2 \) which are in different components of \( X \cap Y \). Let \( P_X(x_1, x_2) \) be the \( x_1x_2 \) path in \( X \), \( P_Y(x_1, x_2) \) the \( x_1x_2 \) path in \( Y \). Since these two paths are not equal, we have

\[
\text{vert}( P_X(x_1, x_2) ) \neq \text{vert}( P_Y(x_1, x_2) ).
\]

Let \( x_4 \in \text{vert}( P_X(x_1, x_2) ) \) \( \bullet \) \( \text{vert}( P_Y(x_1, x_2) ) \).

Using similar definitions, let \( x_3 \in \text{vert}( P_X(x_2, x_1) ) \) \( \bullet \) \( \text{vert}( P_Y(x_2, x_1) ) \).

We again wish to apply the Second Weaving Lemma, Lemma 2.1(b), with \( S = \{ x_1, x_2, x_3, x_4 \} \). Instead of \( v \), we use \( v' \), the word arising from \( v \) by replacing \( x_3 \) by \( x_4 \) wherever \( x_3 \) occurs in context \( x_1x_3x_2 \).

As remarked at the end of Chapter 2 \( v' \) is non-repetitive. Also the only two letter subwords of \( v' \) are \( x_1x_2, x_2x_1, x_1x_4, x_2x_3, x_4x_2, x_3x_1 \). We must now show that we can walk \( v' \) in \( X \cup Y \) modulo paths. There exists an \( x_1x_2 \) path in \( X \cup Y \) not through \( x_4 \), since \( x_4 \) is not on both \( P_X(x_1, x_2) \) and \( P_Y(x_1, x_2) \). Also \( x_3 \) is not on
\[ P_X( x_1, x_2 ) \text{ or } P_Y( x_1, x_2 ) , \text{ because } x_3 \text{ is between } x_2 \text{ and } x_1 \text{ on one of } X \text{ and } Y. \text{ We may thus choose one of } \\
P_X( x_1, x_2 ) \text{ or } P_Y( x_1, x_2 ) \text{ to serve as a path } \\
P( x_1, x_2 ) \text{ having no vertex in } S. \\

Preparing to use the second weaving lemma, with \\
S = \{ x_1, x_2, x_3, x_4 \}, \text{ we have shown that the required } \\
path P( x_1, x_2 ) \text{ exists. Further, since } x_4 \text{ is between } x_1 \text{ and } x_2 \text{ on one of } X \text{ and } Y, \text{ there is an } x_1x_4 \text{ path in } X \cup Y \\
\text{ not through } x_2. \text{ Again } x_3 \text{ is not on this path, for } \\
otherwise } x_3 \text{ is between } x_1 \text{ and } x_4, \text{ hence } x_1 \text{ and } x_2. \\
Arguing similarly, the existence of paths } P( x_1, x_2 ), \\
P( x_1, x_4 ), P( x_4, x_2 ), P( x_2, d ), P( x_2, x_1 ), \\
P( x_3, x_1 ) \text{ may be shown. We may thus walk } v' \text{ on } X \cup Y \\
\text{ modulo paths and therefore } X \cup Y \text{ is versatile. } \\

We have shown that certain digraphs are versatile. \\
We use this intersection lemma to delineate the digraphs \text{ requiring further investigation.} \\

Lemma 3.3 (Classification Lemma): Let } G \text{ be a } \\
strongly connected digraph. Then } G \text{ is of one of the } \\
following types: \\

(1) } \text{vert}( G ) = \text{vert}( X ) \text{ for some directed cycle } X \\
of } G. \text{ In this case, say } G \text{ is a one hump digraph.} \\

(2) } G \text{ is not of type (1), but } \text{vert}( G ) =
vert( X ∪ Y ) where X and Y are directed cycles, and
X ∩ Y is connected and non-empty. In this case, say G is
a two hump digraph.

(3) G is not of types (1) or (2), but
vert( G ) = vert( X ∪ Y ∪ Z ) where X, Y, Z are directed
cycles, X ∩ Y and Y ∩ Z are connected and non-empty, and
X ∩ Z = ø. In this case, say G is a three hump digraph.

(4) G is versatile.

Remark: In fact, unless G falls under one of cases
(1), (2) or (3), G "contains", in a sense to be made
precise later, one of the versatile digraphs MIN.1, MIN.2
or MIN.3.

Proof: If G is versatile, then G falls under case
(4) and we are finished. Thus suppose that G is not
versatile. Since G is strongly connected, write
vert( G ) = ∪_{i=1}^{m} vert( C_i ) where the C_i are directed
cycles of G, and for each j, 2 ≤ j ≤ m,

there exists i < j such that C_j ∩ C_i ≠ ø.

Do this so that m is as small as possible.

If m = 1, then G is of type (1) and we are done. If
m = 2 then G is of type (2), for by the intersection
lemma, since G is not versatile, C_1 ∩ C_2 must be
connected.

If m = 3, C_1 ∩ C_2 ≠ ø. Suppose without loss of
generality that $C_2 \cap C_3 \neq \emptyset$. Otherwise $C_2 \cap C_3 = \emptyset$ so that $C_3 \cap C_1 \neq \emptyset$, and we interchange the roles of $C_1$ and $C_2$.

Because $G$ is not versatile, by the intersection lemma, $C_1 \cap C_2$, $C_2 \cap C_3$ are connected. It remains to show that $C_3 \cap C_1 = \emptyset$. Suppose not.

Let $x_1 \not\in \text{vert}(C_1) \setminus \text{vert}(C_2 \cup C_3)$. Such an $x_1$ exists, for otherwise we could write $\text{vert}(G) = \text{vert}(C_2 \cup C_3)$ where $C_2 \cap C_3$ is non-empty. This contradicts the minimality of $m$.

Similarly we can choose $x_2 \not\in \text{vert}(C_2) \setminus \text{vert}(C_1 \cup C_3)$ and $x_3 \not\in \text{vert}(C_3) \setminus \text{vert}(C_2 \cup C_1)$.

Now we use the second weaving lemma. Let $S = \{x_1, x_2, x_3\}$, and $v = h^{\omega}(x_2)$ as before. The required path $P(x_1, x_2)$ follows $C_1$ from $x_1$ to $C_1 \cap C_2$, then $C_2$ to $x_2$. We see that $x_3$ is not on $P(x, y)$ because $x_3 \not\in C_1 \cup C_2$.

Similarly we can find $P(x_2, x_1)$, $P(x_2, x_3)$, $P(x_3, x_2)$, $P(x_3, x_1)$, $P(x_1, x_3)$. We can walk $v$ on $G$ modulo the $P(x_i, x_j)$, contradicting our assumption that $G$ is not versatile. Here $G$ is cognate, in some sense, to the triangle, MIN.1.

We conclude that if $m = 3$, then $C_1 \cap C_3 = \emptyset$, and $G$ is a three hump digraph.

If $m \geq 4$, we must get a contradiction. We will
consider the cycles $C_1$, $C_2$, $C_3$, $C_4$. As in the previous case, we may assume $C_1 \cap C_2 \neq \emptyset$, $C_2 \cap C_3 \neq \emptyset$, and $C_3 \cap C_1 = \emptyset$.

**Case A: $C_4 \cap C_2 \neq \emptyset$.**

Then pick $x_1 \in \text{vert}(C_4) \setminus \text{vert}(C_1 \cup C_2 \cup C_3)$. We can do this by minimality of $m$. Pick $x_2 \in \text{vert}(C_3) \setminus \text{vert}(C_1 \cup C_2 \cup C_4)$. Such a $x_2$ exists, otherwise $m$ could be reduced by discarding $C_3$. Again, pick $x_3 \in \text{vert}(C_1) \setminus \text{vert}(C_2 \cup C_3 \cup C_4)$.

Again use the second weaving lemma with $S = \{x_1, x_2, x_3\}$ and $v = h^\omega(x_2)$. We can let $P(x_1, x_2)$ be a path from $x_1$ through $C_4$ to $C_4 \cap C_2$, through $C_2$ to $C_2 \cap C_3$, through $C_3$ to $x_2$. Clearly $x_3$ is not on this path.

Similarly we choose $P(x_1, x_3)$, $P(x_2, x_1)$, $P(x_2, x_3)$, $P(x_3, x_1)$, $P(x_3, x_2)$.

By the second weaving lemma, $G$ is versatile, which is a contradiction. (This case is cognate to the undirected graph case where $G$ has a vertex $v$ of degree 3 or greater. Here, $C_2$ plays the role of vertex $v$.)

**Case B: $C_4 \cap C_2 = \emptyset$.**

Suppose without loss of generality that $C_4 \cap C_3 \neq \emptyset$. Otherwise interchange the roles of $C_1$ and $C_3$. Now pick a vertex $1$, with $1 \in \text{vert}(C_1) \setminus \text{vert}(C_2 \cup C_3 \cup C_4)$.

Such a vertex exists because $m$ is minimal. Pick vertex
2 ∈ \text{vert}(C_1 \cap C_2), \text{vertex} 3 ∈ \text{vert}(C_2 \cap C_3), \text{vertex} 4 ∈ \text{vert}(C_3 \cap C_4), \text{and vertex} 5 ∈ \text{vert}(C_4) \setminus \text{vert}(C_1 \cup C_2 \cup C_3). \text{Let} S = \{1, 2, 3, 4, 5\}, \text{and walk} w = g(h^x_2) \text{on G modulo paths, where} h, g \text{are substitutions 2.1 and 2.2 from chapter 2.}

The two letter subwords of v are 12, 23, 34, 45, 54, 43, 32, 21. Choose the paths P(1, 2), P(2, 1) in C_1. Since C_1 \cap C_3 = \emptyset, 3 \text{ and } 4 \text{ are not on } P(1, 2) \text{ or } P(2, 1). \text{Also} 5 \notin C_1 \text{ so that } 5 \text{ is not on } P(1, 2) \text{ or } P(2, 1). \text{Let} P(2, 3) \text{ and } P(3, 2) \text{ be paths in } C_2. \text{These paths avoid } 1 \text{ and } 2 \text{ which are not on } C_2, \text{ and } 4 \text{ and } 5 \text{ which are on } C_4, \text{ as } C_2 \cap C_4 = \emptyset. \text{Choose} P(3, 4), P(4, 3) \text{ in } C_3 \text{ and } P(4, 5), P(5, 4) \text{ in } C_4. \text{By arguments symmetrical to those used with the first four paths, these last four paths satisfy the conditions of the second weaving lemma. Thus } G \text{ is versatile, which is a contradiction. The reader will perceive that we treat } G \text{ as though it were a five element path. (MIN.3) ∎}

The intersection and classification lemmas can be invoked to show that certain classes of digraphs are versatile. To show that an individual digraph is not versatile, it suffices to exhaust the non-repetitive walks on that particular digraph. Next, we provide ways
to show that *classes* of digraphs do not allow arbitrarily long non-repetitive walks.

**Lemma 3.4 (Compressible Paths Lemma):** Let

\[ a_1 a_2 \ldots a_n, \ n \geq 2, \]

be a directed path in a digraph \( G \) with

- \( \text{outdegree}(a_1) = 1, \)
- \( \text{degree}(a_i) = 2, \ i = 2 \text{ to } n-1, \)
- \( \text{indegree}(a_n) = 1. \)

Then \( G \) is versatile if and only if \( G' \) is, where \( G' \) is obtained from \( G \) by removing \( a_2, a_3, \ldots, a_n \), and adding an edge in \( G' \) from \( a_1 \) to every successor of \( a_n \).

(\text{i.e. We identify the vertices of the path.})

**Proof:** The result will follow by induction if we prove the lemma for \( n = 2 \). Suppose then, that \( n = 2 \).

Clearly if \( G' \) is versatile then \( G \) is, by the weaving lemma.

Suppose \( G \) is versatile. Let \( w \) be any non-repetitive walk in \( G \) with the sole restriction that \( w \) does not start with \( a_2 \) or end with \( a_1 \). Consider \( w' = w\vert_{\text{vert}(G)\setminus\{a_2\}} \) the word obtained from \( w \) by deleting all occurrences of \( a_2 \). Clearly \( w' \) will be a walk on \( G' \). If we can show that \( w' \) is non-repetitive, we shall be done, for

\[ |w'| \geq |w|/2, \text{ which can be made arbitrarily large.} \]

If \( v \) is any word on \( \text{vert}(G)\setminus\{a_2\} \), then let \( p(v) \) be the word obtained from \( v \) by replacing each occurrence of \( a_1 \) in \( v \) by \( a_1a_2 \). Then clearly, \( p(w') = w \).
Now suppose for the sake of contradiction that \( w' \) is repetitive, say that \( w' = abbc \) for some \( a, b, c \in \text{vert}(G) \setminus \{a_2\}, b \neq c \). But then \( p(w') = p(a)p(b)p(b)p(c) \), and \( w \) contains a repetition, which is a contradiction.  

**Definition:** Let \( G \) be a digraph so that all the vertices of \( G \) lie on a directed path \( P \) of \( G \). Let \( ij \) be a directed edge of \( G \) not on \( P \). If \( j \) precedes \( i \) in \( P \), then the edge \( ij \) is a **back edge** (with respect to \( P \)). Otherwise, the edge \( ij \) is a **forward edge** (with respect to \( P \)).  

**Definition:** Let \( G \) be a digraph with all its vertices on a directed path \( P \), so that \( \text{vert}(G) \) is ordered. Let \( ij \) be a back edge of \( G \). We say that edge \( ij \) is **useful** if one of the following cases arises:

(i) A forward edge \( kl \) of \( G \) has a vertex between \( j \) and \( i \); \( j \leq l \leq i \) or \( j \leq k \leq i \) (or both).

(ii) There are two back edges of \( G \), \( i'j' \) and \( i''j'' \), such that \( j \leq j' < j'' \leq i' < i'' \leq i \), but not both \( j = j' \) and \( i'' = i \). We say that \( i'j' \) and \( i''j'' \) form an **M** under \( ij \).

(iii) A back edge \( kl \) of \( G \) intersects \( ij \); that is, \( l \leq j \leq k \leq i \) or \( j \leq l \leq i \leq k \). We say that \( kl \) and \( ij \) form an **M**.
Otherwise say that $ij$ is **useless**.

**Lemma 3.5 (M lemma):** Let $G$ be a digraph with all its vertices on a directed path $P$. Let $ij$ be a useless edge of $G$. Then $G$ is versatile if and only if $G \setminus ij$ is, where $G \setminus ij$ is the graph obtained from $G$ by removing the edge $ij$.

**Proof:** First note that removing an edge from $G$ never makes another edge useful.

Next let $Q$ be the set of back edges of $G$ with partial order $\succ : i''j'' \succ i'j'$ if $j'' < j' < i' < i''$, viz. the ends of the smaller edge are between those of the larger.

It suffices to prove the lemma in the case that $ij$ is minimal with respect to this order. Suppose that the lemma has been proved in this case and $kl$ is any useless edge of $G$. Let the set of useless edges of $G$ less than or equal to $kl$ be $S = \{ i_1j_1, i_2j_2, \ldots, i_nj_n, kl \}$. Then $G \setminus S$ is versatile if and only if $G$ is; we simply remove the edges of $S$ from $G$ one at a time, at each step removing a minimal edge. To get $G \setminus kl$, we add the edges of $S \setminus \{ kl \}$ to $G \setminus S$, starting with maximals.

Suppose then that $ij$ is a useless edge of $G$, minimal in the order given. Let $v$ be a non-repetitive word of type
ω walkable on G. If ij appears only finitely often in v, then a final segment of v can be walked on G \ ij, and we are done. Thus assume that ij appears infinitely often as a subword in v. We can then find arbitrarily long subwords w of v such that w has i as a suffix.

**Claim:** Any long enough subword w of v having i as a suffix must have suffix j(j+1)(j+2)...(i-1)i. (Here j+1 is the successor of j on P etc.)

**Proof of Claim:** The indegree of i is 1: Any forward edge ending at i satisfies (i) of the definition of useful edges, making ij useful. Any back edge ending at i satisfies (iii) of the definition, making ij useful.

Thus w ends in (i-1)i.

Now suppose that long enough w ending in i must end in

(i-k)(i-k+1)...(i-1)i, j < i-k  \( \text{......}(*) \)

We show that w ends in (i-k-1)(i-k)...(i-1)i.

Suppose not. Then some edge e = l(i-k), l ≠ i-k-1 exists in G, and w ends in l(i-k)...i. If l < i-k, then e is a forward edge satisfying (i) of the definition of useful edges, a contradiction.

Thus we must assume that e is a back edge. Because of (iii) of the definition of useful edges, we must have l ≤ i. Since e is not a useless edge, by the minimality
of $ij$, there are two possibilities:

I) There is an $M$ under $e$. Such an $M$ is also under $ij$, a contradiction, as per (ii) of the definition of useful edges.

II) Some edge $f = rs$ forms an $M$ with $e$ where

$s < i-k < r < l$

or

$i-k < s < l < r$.

Because of (ii), (iii) of the definition of useful edges, we insist that $j = s < i-k < r < l = i$

or

$j = i-k < s < l < r = i$.

However by assumption, $j < i-k$, so we must have

$j = s < i-k < r < l = i$.

Thus $w$ ends in $l(i-k) ... i = i(i-k)i$. But then, if $w$ is long enough, our induction hypothesis (* says that $w$ ends in $(i-k) ... i(i-k) ... i$, and $v$ contains a repetition, which is a contradiction.

Thus $w$ ends in $(i-k-1)(i-k) ... i$. By induction, $w$ ends in $j(j+1)(i-1)i$.\[\square\]

A second claim has a similar proof.

**Claim:** Any long enough subword $w$ of $v$ having $j$ as a prefix must have prefix $j(j+1)(j+2) ... (i-1)i$.

However $v$ contains $ij$ infinitely often, so that we can find a long subword $wijz$ of $v$ with $w = w'j(j+1)(j+2) ... (i-1)i$, $z = j(j+1)(j+2) ... (i-1)iz'$. But
then $v$ contains the repetitive subword

$$w'j(j+1)(j+2)...(i-l)ij(j+1)(j+2)...(i-1)iz',$$

a contradiction. We conclude that $v$ contains $ij$ only finitely often, and thus $G \setminus ij$ is versatile if and only if $G$ is.

Clearly the existence of a loop in a digraph does not help to make it versatile. We may therefore modify Lemma 3.4 slightly:

**Lemma 3.6 (Compressible Paths Lemma):** Let $a_1 a_2 ... a_n$ be a directed path in a digraph $G$ with

- $\text{outdegree}(a_1) = 1$,
- $\text{degree}(a_i) = 2$, $i = 2$ to $n-1$,
- $\text{indegree}(a_n) = 1$.

Then $G$ is versatile if and only if $G'$ is, where $G'$ is obtained from $G$ by removing $a_2, a_3, ..., a_n$ and adding an edge in $G'$ from $a_1$ to every successor of $a_n$ other than $a_1$.

We say that digraph $G$ reduces to digraph $H$ ($H$ is a reduction of $G$) if $H$ is obtained from $G$ by repeated applications of the compressible paths lemma and removal of loops and useless edges. Thus if $G$ reduces to $H$, $G$ is versatile if and only if $H$ is versatile.
The purpose of this thesis is to characterize versatile digraphs. We make this characterization by producing two sets of digraphs, MIN (shown in Appendix 1) and MAX (shown in Appendix 2). In Chapters 7 and 8 we show that the digraphs of MIN are versatile. In Chapter 9, we show that the digraphs of MAX are not versatile. In Chapter 4, Chapter 5 and Chapter 6, the heart of the thesis, we give a case by case breakdown of all digraphs to show that every digraph either can be reduced to some digraph "contained" in a digraph of MAX, and hence is non-versatile, or else "contains" some digraph of MIN, and hence is versatile. The intersection lemma, the classification lemma, the M lemma, and the definitions of useless edges, forward edges and back edges will be used to give this case breakdown of digraphs. The next section of this chapter introduces the concept of mimicking, by which we make precise what it means for a digraph G to "contain" a digraph H.

**Definition:** Let H, G be digraphs so that there is an injection \( m: \text{vert } H \rightarrow \text{vert } G \), such that whenever \( ij \) is an edge of H, then there is a path in \( G \setminus m(\text{vert } H) \) from \( m(i) \) to \( m(j) \). We say that G *imitates* H.

We can put this another way: We fix a labelling of G. Whenever \( v \) is a walk on H then \( v \) can be walked on G.
modulo paths with respect to this labelling. It follows that if $G$ imitates $H$, then if $H$ is versatile, so is $G$.

Example: The graph of Figure 3.1 imitates the triangle with the given labelling.

Not every versatile digraph imitates $P_5$ or the triangle. (Otherwise we would be finished, by Chapter 2.) The digraph $G$ of Figure 3.2 is a counterexample. This graph is indeed versatile, because the following substitution is suitable.

$g: x_1 \rightarrow 1232$

$x_2 \rightarrow 123454$

$x_3 \rightarrow 123456$

This is easy to check, or refer to the Different Endings Lemma of Chapter 7. However, an argument could be given to show that $G$ can imitate neither the triangle nor the five element path.

If $G$ is a digraph, then $G^R$, the reverse of $G$, is the digraph with the same vertex set as $G$, and a directed edge $ij$ exactly when $ji$ is a directed edge of $G$. Clearly $G^R$ is versatile if and only if $G$ is. To reduce the size of $MIN$, we have sought to include at most one of $G$ and $G^R$ for any digraph $G$. Let us extend the idea of imitation to take advantage of this:

Definition: Let $H, G$ be digraphs. Say that $G$ mimics

}\]
Figure 3.1

Figure 3.2
H if G imitates at least one of $H, H^R$.

Now that we have introduced the concept of mimicking, we remark that the proofs of Lemmas 3.2 and 3.3 prove the following stronger results:

Lemma 3.2' (Intersection Lemma): Let $X, Y$ be directed cycles in the digraph $G$ so that $\text{vert}(X) \cap \text{vert}(Y) \neq \emptyset$. Then either

1) $X \cap Y$ is connected

or 2) $X \cup Y$ mimics one of MIN.1 or MIN.2, and hence is versatile

Lemma 3.3' (Classification Lemma): Let $G$ be a strongly connected digraph. Then $G$ is of one of the following types:

1) $\text{vert}(G) = \text{vert}(X)$ for some directed cycle $X$ of $G$. In this case, say $G$ is a one hump digraph.

2) $G$ is not of type (1), but $\text{vert}(G) = \text{vert}(X \cup Y)$ where $X$ and $Y$ are directed cycles, and $X \cap Y$ is connected and non-empty. In this case, say $G$ is a two hump digraph.

3) $G$ is not of types (1) or (2), but $\text{vert}(G) = \text{vert}(X \cup Y \cup Z)$ where $X, Y, Z$ are directed cycles, $X \cap Y$ and $Y \cap Z$ are connected and non-empty, and
In this case, say $G$ is a **three-hump** digraph.

(4) $G$ mimics one of MIN.1, MIN.2 or MIN.3, and therefore is versatile.

We now have the tools necessary to state and prove our main result. The main theorem of this work is proved in three pieces, appearing in Chapter 4, Chapter 5 and Chapter 6, respectively.

**Theorem 3.8:** Let $G$ be a three hump digraph. Either $G$ mimics a graph $H$ where $H$ is in MIN, or a reduction of $G$ is mimicked by some digraph $K$, where $K$ is in MAX.

**Theorem 3.9:** Let $G$ be a two hump digraph. Either $G$ mimics a graph $H$ where $H$ is in MIN, or a reduction of $G$ is mimicked by some digraph $K$, where $K$ is in MAX.

**Theorem 3.10:** Let $G$ be a one hump digraph. Either $G$ mimics a graph $H$ where $H$ is in MIN, or a reduction of $G$ is mimicked by some digraph $K$, where $K$ is in MAX.

**The Main Theorem (Theorem 3.11):** Let $G$ be a digraph. Either $G$ mimics a graph $H$ where $H$ is in MIN, or a reduction of $G$ is mimicked by some digraph $K$, where $K$
is in MAX.

**Corollary 3.12:** Let $G$ be any digraph. Either $G$ is non-versatile, or else $G$ mimics a graph $H$ in MIN.
Chapter 4: Three Hump Digraphs

In this chapter we prove Theorem 3.8.

Theorem 3.8: Let $G$ be a three hump digraph. Either $G$ mimics a graph $H$ where $H$ is in MIN, or a reduction of $G$ is mimicked by some digraph $K$, where $K$ is in MAX.

We begin by proving a refinement of the classification lemma.

Lemma 4.1 (Refining the Classification Lemma): Let $G$ be a three hump digraph. Then either

1. $\text{vert}(G) = \text{vert}(X \cup Y \cup Z)$ where $X, Y, Z$ are cycles of $G$, $X \cap Y, Y \cap Z$ are connected and non-empty, $X \cap Z = \emptyset$, and $Y \setminus (X \cup Z)$ is connected.

or


Proof: Suppose that $Y \setminus (X \cup Z)$ is not connected. Then choose vertices $1 \in X \setminus Y, 2 \in X \cap Y, 4 \in Y \cap Z$ and $5 \in Z \setminus Y$. Pick two vertices $3, 3'$ from different components of $Y \setminus (X \cup Z)$. Without loss of generality we may assume that vertices $2, 3, 3', 4$ appear in cyclical order $2, 3, 4, 3'$ in $Y$. (Recall that $X \cap Y, Z \cap Y$ are connected.) With this labelling, $G$ mimics MIN.4. (See Figure 4.1.)

This refinement of the classification lemma allows us to
Figure 4.1
introduce a certain structure to three hump digraphs.

**Definition:** Let G be a three hump digraph. We say that G has a skeleton if

1. We can write \( \text{vert}(G) = \text{vert}(P) \) where P is a directed Hamiltonian path in G. Path P gives an order to the vertices of G.

2. With respect to this order, G has at least three additional edges \( a_2 a_1, b_2 b_1, c_2 c_1 \) where \( a_1 < b_1 \leq a_2 < c_1 \leq b_2 < c_2 \), \( a_1 \) is the initial vertex of P, \( c_2 \) the final vertex.

We call the digraph made up of P together with the edges \( a_2 a_1, b_2 b_1, c_2 c_1 \) the skeleton of G. (See Fig. 4.2.) Other edges of G are called extra-skeletal edges.

**Lemma 4.2 (The Skeleton):** Let G be a three hump digraph which does not mimic MIN.4. Then G has a skeleton.

**Proof:** We may assume by Lemma 4.1 that \( Y \setminus (X \cup Z) \) is a directed path. Let \( m \) be the source of this directed path and \( M \) the sink. Let \( a_2 \) be the predecessor of \( m \) in \( Y \). Either \( a_2 \in X \) or \( a_2 \in Z \), but not both. Suppose without loss of generality (up to renaming) that \( a_2 \in X \). Let \( a_1 \) be the successor of \( a_2 \) in X.
Let $c_1$ be the successor of $M$ in $Y$. Then $c_1 \in Z$. Otherwise $c_1 \in X$, and $Y$ is the union of two directed paths: the segment of $Y$ from $m$ through $M$, and the segment of $X$ from $c_1$ through $a_2$. (Recall that $Y \cap X$ is connected.) But then $Y \cap Z = \emptyset$, since $X \cap Z = \emptyset$ and $(Y \setminus (X \cup Z)) \cap Z = \emptyset$. This is a contradiction. Thus indeed $c_1 \in Z$. Let the predecessor of $c_1$ in $Z$ be $c_2$.

Now $X \cap Y$ is a directed path with $a_2$ as sink. Let $b_1$ be the source. Let $b_2$ be the sink of the directed path $Y \cap Z$, which has $c_1$ as source. (See Figure 4.3.)

Now $X$ is the cycle $a_1 \rightarrow a_2$, $Y \setminus (X \cup Z)$ is the path $m \rightarrow M$, and $Z$ is the cycle $c_1 \rightarrow c_2$. We may therefore let $P = a_1 \rightarrow a_2 \rightarrow m \rightarrow M \rightarrow c_1 \rightarrow c_2$. Clearly we have $\text{vert}(G) = \text{vert}(P)$,

$$a_1 \leq b_1 \leq a_2 \leq c_1 \leq b_2 \leq c_2,$$

$a_1$ is the initial vertex of $P$, $c_2$ the final vertex of $P$.

We show that $a_1 < b_1$, $a_2 < c_1$, $b_2 < c_2$.

If $a_1 = b_1$, then $X = a_1 \rightarrow a_2 = b_1 \rightarrow a_2 \subset X \cap Y$, so that $X \subset Y$. Then $\text{vert}(G) = \text{vert}(Y \cup Z)$, a contradiction.

Similarly $b_2 \neq c_2$.

Finally, $a_2 \neq c_1$, as $X \cap Z = \emptyset$.

The edges $a_2a_1$, $c_2c_1$ exist by definition. The edge
Proof of Theorem 3.8: We assume that G has a skeleton, since otherwise, by Lemma 4.2 above, G mimics MIN.4. Also, we may assume that G has no useless edges, as such edges may be removed without affecting whether G is versatile or not. The proof of the theorem involves a lengthy enumeration of cases. To make this case breakdown we refer to the skeleton of G (Figure 4.2). Let the extra-skeletal edges of G be \( i_1 j_1, i_2 j_2, \ldots, i_m j_m \). We make cases based on m.

To reduce work, we often invoke symmetry. Now \( G^R \), the reverse of G is a three hump digraph. Again, \( \text{vert}(G^R) = \text{vert}(P^R) \) where \( P^R \) is the reverse of P. Renaming \( a_1 \) as \( c_2' \), \( a_2 \) as \( c_1' \), \( b_1 \) as \( b_2' \), \( b_2 \) as \( b_1' \), \( c_1 \) as \( a_2' \) and \( c_2 \) as \( a_1' \), we see that \( G^R \) is a three hump digraph with skeleton \( P^R \cup \{ c_2'c_1', b_2'b_1', a_2'a_1' \} \). This symmetry under reversal reduces the number of required cases. For example, suppose G has an edge \( i_1 j_1 \) with \( i_1 > b_2 \). Then, renaming \( i_1 \) as \( j_1' \) and \( j_1 \) as \( i_1' \), the reversal of G has \( j_1' < b_1 \) and will later fall under our case A1. Keeping this use of symmetry in mind, we proceed to our case division.

\( m = 0 \): If G is its own skeleton then a reduction of G can be mimicked on MAX.1 and we are done. (See Figure
4.2. We apply the Compressible Paths Lemma, Lemma 3.6, to the paths in $G$

- from $a_1$ to the predecessor in $P$ of $b_1$
- from $b_1$ to $a_2$
- from the successor of $a_2$ to the predecessor of $c_1$
- from $c_1$ to $b_2$
- and from $b_2$ to $c_2$.

The result is isomorphic to a graph in one of Figure 4.4 or Figure 4.5, depending on whether there is a vertex between $a_2$ and $c_1$ in $G$. These graphs are mimicked by $\text{MAX.1}$ with the given labellings.

$m = 1$: Depending on $i_1$, $j_1$ we have several subcases.

Case A: The edge $i_1j_1$ is a back edge; i.e. With respect to the order given to $\text{vert}(G)$ by $P$, $i_1 > j_1$.

Case B: The edge $i_1j_1$ is a forward edge; i.e. $i_1 < j_1$.

Case A breaks down as follows:

Case A1 $a_1 \leq j_1 < b_1$ (or symmetrically, $i_1 > b_2$).

Case A2 $b_1 \leq j_1 < a_2$ (and $i_1 \leq b_2$).

Case A3 $a_2 < j_1 < c_1$ (and $i_1 < c_1$).

In this third case, the edge $i_1j_1$ is useless, a contradiction. (See Figure 4.6)
Figure 4.4

Figure 4.5

Figure 4.6
Cases A1 and A2 are further subdivided.

**A1:**

*Case A1 (a)*
\[a_1 \preceq j_1 \prec i_1 \prec a_2.\]

Here the edge \(i_1j_1\) is useless, a contradiction. (See Figure 4.7.)

*Case A1 (b)*
\[a_1 \preceq j_1 \prec b_1 \preceq i_1 \prec a_2.\]

Here \(G\) mimics MIN.5. (See Figure 4.8.) The labelling of vertices of \(G\) required by the definition of mimicking is shown explicitly in the figure.

*Case A1 (c)*
\[a_1 \preceq j_1 \prec b_1 \preceq a_2 \prec i_1 \prec c_1.\]

If \(a_2 = i_1\), then \(a_1 < j_1\) so that \(G\) mimics MIN.6. (See Figure 4.9.)

If \(a_2 < i_1\), then \(G\) mimics MIN.5. (See Figure 4.10.)

*Case A1 (d)*
\[a_1 \preceq j_1 \prec b_1 \prec c_1 \preceq i_1\]

Note that \(i_1 < c_2\) and \(a_1 < j_1\) or \(\text{vert}(G)\) could be written as the union of two cycles.

Here \(G\) mimics MIN.7. (See Figure 4.11.)

**A2:**

*Case A2 (a)*
\[b_1 \preceq j_1 \preceq i_1 \preceq a_2.\]

Here the edge \(i_1j_1\) is useless, a contradiction. (See Figure 4.12.)
Figure 4.7

Figure 4.8

Figure 4.9
Case A2 (b) \( a_2 < i_1 < c_1 \).

If \( j_1 = b_1 \) then the reduction of \( G \) can be mimicked on MAX.1 (See Figure 4.13.) and we are done.

If \( j_1 > b_1 \) then \( G \) mimics MIN.8. (See Figure 4.14.)

Case A2 (c) \( c_1 \leq i_1 \leq b_2 \).

Either \( j_1 \neq b_1 \) or \( i_1 \neq b_2 \), since \( i_1 j_1 \neq b_2 b_1 \). Since \( b_1 \leq j_1 \leq a_2 \) and \( c_1 \leq i_1 \leq b_2 \), the roles of \( i_1 \) and \( j_1 \) are reversed when \( G \) is reversed. (See Figure 4.15.) Therefore, without loss of generality, suppose that \( j_1 \neq b_1 \). This takes us from Figure 4.15 to Figure 4.16.

But now \( a_1 < j_1 \leq a_2 < c_1 \leq i_1 < c_2 \), and \( a_1 < b_1 < j_1 \). Thus \( j_1 \) and \( i_1 \) can play the roles of \( b_1 \) and \( b_2 \) in the skeleton of \( G \). Switching the roles of \( i_1 j_1 \) and \( b_2 b_1 \) gives case A1 (d) which has already been dealt with.

This concludes case A.

Case B is divided as follows:

Case B1: \( i_1 < b_1 \).

Case B2: \( b_1 \leq i_1 \leq a_2 \ (j_1 \leq b_2) \).

Case B3: \( a_2 < i_1 < j_1 < c_1 \).

In Case B3, let \( x \) be a vertex between \( i_1 \) and \( j_1 \) on
P. Then G mimics MIN.2. (See Figure 4.17.)

Cases B1 and B2 are further subdivided.

B1: Case B1 (a): \( j_1 < b_1 \)
Let \( x \) be a vertex between \( i_1 \) and \( j_1 \) on \( R \). Here G mimics MIN.2. (See Figure 4.18.)

Case B1 (b): \( j_1 = b_1 \).
Here G mimics MIN.9. (See Figure 4.19.)

Case B1 (c): \( b_1 < j_1 < a_2 \).
Here G mimics MIN.10. (See Figure 4.20.)

Case B1 (d): \( a_2 < j_1 < b_2 \).
Consider the cycle \( C \) following \( P \) from \( a_1 \) to \( i_1 \), edge \( i_1 j_1 \), \( P \) from \( j_1 \) to \( b_2 \), edge \( b_2 b_1 \), \( P \) from \( b_1 \) to \( a_2 \), then edge \( a_2 a_1 \). Recall the cycle \( Z \) from the proof of Lemma 4.2: \( Z \) is the cycle consisting of the path in \( P \) from \( c_1 \) to \( c_2 \), together with the edge \( c_2 c_1 \). We see that \( C \cap Z \) is connected. Therefore, \( \text{vert}(G) \neq \text{vert}(C \cup Z) \), as \( G \) is a three hump digraph. We have two possibilities:

(i) There is a vertex \( x \) between \( i_1 \) and \( b_1 \)
Here G mimics MIN.11. (See Figure 4.21.)
Figure 4.18

Figure 4.19

Figure 4.20

Figure 4.21
(ii) There is a vertex \( x \) between \( a_2 \) and the lesser of \( j_1, c_1 \).
Here \( G \) mimics MIN.12. (See Figure 4.22.)

Case B1 (c): \( b_2 < j_1 \)
Consider the cycle \( C \) following \( P \) from \( a_1 \) to \( i_1 \), then edge \( i_1j_1 \), then \( P \) from \( j_1 \) to \( c_2 \), then edge \( c_2c_1 \), then \( P \) from \( c_1 \) to \( b_2 \), then edge \( b_2b_1 \), then \( P \) from \( b_1 \) to \( a_2 \), then edge \( a_2a_1 \). Now \( \text{vert}(G) \) cannot equal \( \text{vert}(C \cup X) \), \( \text{vert}(C \cup Y) \) or \( \text{vert}(C \cup Z) \). This forces one of two cases:

(i) There is a vertex \( x \) of \( P \) between \( a_2 \) and \( c_1 \).
Here \( G \) mimics MIN.1. (See Figure 4.23.)

(ii) There are vertices of \( P \) between \( i_1 \) and \( b_1 \) and between \( b_2 \) and \( j_1 \).
Here \( G \) mimics MIN.13. (See Figure 4.24.)

B2:

Case B2 (a) \( j_1 < c_1 \).
Here \( G \) mimics MIN.2. (See Figure 4.25. Let \( x \) be any vertex between \( i_1 \) and \( j_1 \).)

Case B2 (b) \( c_1 \leq j_1 \leq b_2 \).
We make two cases:

(i) There is some vertex \( x \) of \( G \), \( a_2 < x < c_1 \). Here \( G \)
mimics MIN.1. (See Figure 4.26.)

(ii) There is no vertex of $G$ between $a_2$ and $c_1$ on $P$. Then we cannot have $i_1 = a_2$ and $j_1 = c_1$, as $i_1 j_1$ was chosen to be an extra-skeletal edge. By the symmetry of this case under reversal, we may assume that $i_1 \neq a_2$.

Replace $P$ by the hamiltonian path $H$. $H$ starts with the successor of $i_1$, follows $P$ to $a_2$, then follows edge $a_2 a_1$ to get to $a_1$. Then $H$ follows $P$ from $a_1$ to $i_1$, then $i_1 j_1$ to $j_1$. Next, $H$ follows $P$ from $j_1$ to $c_2$. If $c_1 = j_1$, then $H$ stops at $c_2$. Otherwise, $H$ follows edge $c_2 c_1$ to $c_1$, then $P$ to the predecessor of $j_1$. (See Figure 4.27.) With respect to the new skeleton, $G$ falls under case B1(e), which has already been dealt with.

This completes the case when $m = 1$.

$m > 1$:

Without loss of generality we can assume that edge $i_1 j_1$ falls (up to reversal of $G$) under one of cases A1(a), A2(a), A2(b) or A3 of the classification for $m = 1$. This is true because we have shown that if $G$ contains an edge $i_1 j_1$ falling under one of the other cases, $G$ mimics a graph of MIN. Likewise assume that every other extra-skeletal edge of $G$ falls under one of these cases.
Figure 4.27
(Under the appropriate renaming, of course.) We thus use these cases for the breakdown of the present case.

Case A1 (a) \[a_1 \leq j_1 < i_1 < b_1\].
Suppose that edge \(i_1j_1\) falls under case A1 (a). Since \(i_1j_1\) is not a useless edge, and \(G\) has no forward edges, either there is an M under edge \(i_1j_1\), or an edge forms an M with edge \(i_1j_1\). However, any edge under \(i_1j_1\) is an edge of case A1 (a). Likewise, of the four types of edges remaining, only those falling under case A1 (a) could form an M with \(i_1j_1\). Thus without loss of generality (up to renaming), say that edges \(i_1j_1\) and \(i_2j_2\) form an M, with \(a_1 \leq j_1 < j_2 \leq i_1 < i_2 < b_1\). Here \(G\) mimics MIN.14. (See Figure 4.28.)

Case A2 (a) \[b_1 \leq j_1 < i_1 < a_2\].
Without loss of generality (up to renaming), edges \(i_1j_1\) and \(i_2j_2\) form an M, \(b_1 \leq j_1 < j_2 \leq i_1 < i_2 \leq a_2\). Here \(G\) mimics MIN.15. (See Figure 4.29.)

Case A3 \[a_2 \leq j_1 < i_1 < c_1\].
Without loss of generality (up to renaming), edges \(i_1j_1\) and \(i_2j_2\) form an M. However, we now have two possibilities:
(i) Edge $i_2j_2$ falls under case A3 (after appropriate renaming.) Without loss of generality (up to renaming),
\[ a_2 < j_1 < j_2 < i_1 < i_2 < c_1. \]
Here $G$ mimics MIN.15. (See Figure 4.30.)

(ii) Edge $i_2j_2$ falls under case A2 (b) (after appropriate renaming.) Without loss of generality (up to renaming),
\[ b_1 = j_2 < a_2 < j_1 < i_2 < i_1 < c_1. \]
Here $G$ mimics MIN.16. (See Figure 4.31.)

Case A2 (b) \[ b_1 = j_1 < a_2 < i_1 < c_1. \]
Without loss of generality, we may now assume that every extra-skeletal edge of $G$ falls under case A2(b). However with reversals, this allows three possibilities:

(i) We have $b_1 = j_2$, $i_1 < i_2 < c_1$. Here $G$ mimics MIN.5.
(See Figure 4.32.)

(ii) We have $b_2 = i_2$, $j_2 < i_1$. Here $G$ mimics MIN.3.
(See Figure 4.33.)

(iii) We have $b_2 = i_2$, $j_2 > i_1$. Here $G$ mimics
MIN.17. (See Figure 4.34.)

We have now proved the theorem.
Chapter 5: Two Hump Digraphs

In this chapter we will consider two hump digraphs.

We prove

**Theorem 3.9:** Let \( G \) be a two hump digraph. Then either \( G \) mimics a digraph \( H \) in MIN, or a reduction of \( G \) is mimicked by a digraph \( K \) in MAX.

In analogy to the previous chapter, we introduce skeletons.

**Definition:** Let \( G \) be a two hump digraph. Then we say that \( G \) has a skeleton if

1. We can write \( \text{vert}(G) = \text{vert}(P) \) where \( P \) is a directed Hamiltonian path in \( G \).
2. \( G \) has at least two additional edges \( a_2a_1, b_2b_1 \) where \( a_1 < b_1 < a_2 < b_2 \) with respect to the order \( P \) induces on \( \text{vert}(G) \). \( a_1 \) is the initial vertex of \( P \), \( b_2 \) the terminal vertex of \( P \).

We call the digraph made up of \( P \) together with the edges \( a_2a_1, b_2b_1 \) the skeleton of \( G \). (See Figure 5.1) Other edges of \( G \) are called extra-skeletal edges.

**Lemma 5.1 (The Skeleton):** Let \( G \) be a two hump digraph. Then \( G \) has a skeleton.
Proof: We know that \( Y \cap X \) is a directed path. Let \( b_1 \) be the source of this directed path and \( a_2 \) the sink. Let \( b_2 \) be the predecessor of \( b_1 \) in \( Y \). Let \( a_1 \) be the successor of \( a_2 \) in \( X \). (See Figure 5.2)

Then the vertices of \( G \) all lie on the directed path \( a_1 \rightarrow b_1 \rightarrow a_2 \rightarrow b_2 \). The edges \( a_2a_1, b_2b_1 \) exist by definition. Finally, in the case that \( a_1 = b_1 \) or \( a_2 = b_2 \), \( \text{vert}(G) \) lies on a cycle, \( X \) or \( Y \) respectively.

Remark: The roles of \( X \) and \( Y \) in the previous proof are interchangeable.

Proof of Theorem 3.9: This proof involves a very long enumeration of cases, classifying the two hump digraphs. Assume again that \( G \) has no useless edges. Again the case breakdown refers to the skeleton of \( G \). Label the extra-skeletal edges of \( G \) by \( i_1j_1, i_2j_2, \ldots, i_mj_m \). We make cases based on \( m \).

\( m = 0 \): If \( G \) is its own skeleton we are done. Here a reduction of \( G \) can be mimicked by MAX.1. (See Figure 5.1)

\( m = 1 \): We have two branches to our case division:

Case I: The edge \( i_1j_1 \) is a back edge; i.e. \( i_1 > j_1 \) with respect to the order given by \( P \).

Case II: The edge \( i_1j_1 \) is a forward edge; i.e.
Figure 5.2
i_1 < j_1.

**CASE I (ONE BACK EDGE)**

Case I gives rise to several subcases. If $a_1 < j_1 < b_1$, then we form cases based on $i_1$.

**Case A**

$a_1 < j_1 < b_1$ and $i_1 < b_1$.

Here the edge $i_1j_1$ is useless, a contradiction. (See Figure 5.3)

**Note:** Later on in the proof, when we consider the possibility that $m > 1$, it will be useful to have names for the various types of back edges occurring in $G$. When $m = 1$, we have 5 subcases of case I, viz. cases A, B, C, D and E. We call an edge $i_rj_r$ of $G$ a **type A** (B, C, D, E) edge if the graph $G'$, formed by removing from $G$ all extra-skeletal edges other than $i_rj_r$, falls under case A (B, C, D, E) of the present discussion.

**Case B**

$a_1 = j_1$ and $b_1 \leq i_1 \leq a_2$.

(Therefore $i_1 \neq a_2$.)

A reduction of $G$ can be mimicked by MAX.12. (See Figure 5.4)
Figure 5.3

Figure 5.4
Case C \( a_1 < j_1 < b_1 \) and \( b_1 < i_1 < a_2 \).

A reduction of \( G \) can be mimicked by MAX.2. (See Figure 5.5)

In the next case we will want to invoke symmetry. In preparation, we note that a digraph \( G \) is versatile if and only if \( G^R \), the reverse of \( G \), is versatile. It is useful now to extend our concept of type A, B, C, D, E edges to reverse edges. An edge \( i_rj_r \) is a reverse type A (B, C, D, E) edge if \( G'^R \) falls under case A (B, C, D, E) of the present discussion, where \( G' \) is again the graph formed from \( G \) by removing all extra-skeletal edges other than \( i_rj_r \).

Case D \( a_1 < j_1 < b_1 \) and \( a_2 < i_1 \).

If \( a_1 = j_1 \), then \( i_1 < b_2 \), otherwise the edge \( i_1j_1 \) taken with the path \( P \) forms a cycle through all the vertices of \( G \), a contradiction. However now \( i_1j_1 \) and \( a_2a_1 \) can interchange roles, and we are in case B. Thus in the present case, we assume without loss of generality that \( a_1 < j_1 \). Symmetrically, we assume that \( i_1 < b_2 \), or \( G \) is the reverse of a graph falling under case B. A reduction of \( G \) can be mimicked by MAX.15. (See Figure 5.6)
Figure 5.5

Figure 5.6
This concludes an enumeration of the subcases when $a_1 \leq j_1 < b_1$.

To reduce work, we again invoke symmetry. Let $G$ have an edge $i_1j_1$ with $a_2 < i_1 \leq b_2$. Then $G^R$, the reverse of $G$, is clearly a two hump digraph. Again, $\text{vert}(G^R) = \text{vert}(P^R)$ where $P^R$ is the reverse of $P$. Renaming $a_1$ as $b_2'$, $a_2$ as $b_1'$, $b_1$ as $a_2'$, $b_2'$ as $a_1'$, $i_1$ as $j_1'$ and $j_1$ as $i_1'$, we see that $G^R$ is a two hump digraph with skeleton $P^R \cup \{ b_2'b_1', a_2'a_1' \}$ and an additional edge $i_1'j_1'$ with $a_1' \leq j_1' < b_1'$. We see that the case when $a_2 < i_1 \leq b_2$ and the case when $a_1 \leq j_1 < b_1$ are symmetric and may be regarded as equivalent. For $m > 1$, however, it will occasionally be necessary to distinguish between "normal" type B or C edges, and "reversed" type B or C edges.

Case E $b_1 \leq j_1 \leq a_2$ (and $i_1 \leq a_2$).

Here the edge $i_1j_1$ is useless, a contradiction. (See Figure 5.7)

Thus when $m = 1$, and $i_1j_1$ is a back edge, $G$ is mimicked by one of the graphs of MAX. We also draw attention to the 5 basic types of back edge which $G$ can have. These 5 types of edges figure in later case.
Figure 5.7
CASE II (ONE FORWARD EDGE)

Case II also gives rise to several subcases; however, this case may be dealt with very simply once we have made the following observation: For most forward edges $ij$, a skeleton for $G$ can be chosen so that $ij$ becomes a back edge with respect to that skeleton.

As we remarked earlier, there is a skeleton for $G$ in which the roles of $X$ and $Y$ are reversed. Now the vertices of $G$ may be divided into three sets:

$$\text{vert}(X \setminus Y), \text{vert}(X \cap Y), \text{vert}(Y \setminus X).$$

With respect to the skeleton we have given for $G$, these sets occur in this order. However, if the roles of $X$ and $Y$ are reversed, then the order of these sets reverses.

Thus if $i_1$ and $j_1$ are not both in the same one of these sets, a forward edge $i_1j_1$ becomes a back edge when $X$ and $Y$ are interchanged. Therefore in the present case we assume without loss of generality that $i_1$ and $j_1$ are both in the same one of the listed sets.

Case II.1: We have $i_1, j_1 \in X \setminus Y$, i.e.

$$a_1 \leq i_1 < j_1 < b_1.$$

Some vertex $x$ of $P$ must lie between $i_1$ and $j_1$. Here $G$
mimics MIN.2 (See Figure 5.8)

Case II.2: We have \( i_1, j_1 \in X \cap Y \), i.e.,
\[ b_1 \leq i_1 < j_1 \leq a_2. \]
Consider the cycle \( C \) differing from \( X \) in that the path from \( i_1 \) to \( j_1 \) in \( X \) is replaced in \( C \) by the edge \( i_1 j_1 \).
(See Figure 5.9.) Then cycles \( C \) and \( Y \) have an intersection which is not connected, and by the proof of the Intersection Lemma, Lemma 3.2, \( G \) mimics MIN.1 or MIN.2.

Case II.3: We have \( i_1, j_1 \notin Y \setminus X \).
Interchanging the roles of \( X \) and \( Y \), this case is equivalent to case II.1.

This concludes CASE II, and hence the case when \( m = 1 \).

\( m > 1 \):

In light of the foregoing, we may now assume that any individual forward edge of \( G \) may be turned into a back edge by interchanging cycles \( X \) and \( Y \). We will now show that in fact all extra-skeletal edges of \( G \) may simultaneously be assumed to be back edges. Suppose that \( i_1 j_1 \) is a back edge and \( i_2 j_2 \) is a forward edge. By Case II, we may assume that reversing the roles of \( X \) and \( Y \)
Figure 5.8

Figure 5.9
would make \( i_2 j_2 \) a back edge. We make cases as follows.

Case II+A: Edge \( i_1 j_1 \) falls under case A. By assumption, edge \( i_2 j_2 \) becomes a back edge when the roles of X and Y are interchanged. However, in the present case, when X and Y are switched, \( i_1 j_1 \) remains a back edge (since \( i_1, j_1 \in X \)). We may thus assume that both \( i_1 j_1 \) and \( i_2 j_2 \) are back edges.

Case II+B: Edge \( i_1 j_1 \) falls under case B.

Case II+C: Edge \( i_1 j_1 \) falls under case C.

Case II+D: Edge \( i_1 j_1 \) falls under case D.

Case II+E: Edge \( i_1 j_1 \) falls under case E. In this case, if the roles of X and Y are interchanged, \( i_1 j_1 \) and \( i_2 j_2 \) are both back edges and we are finished. (Both \( i_1, j_1 \in X \cap Y \).

In case II+B we make the following subcases based on \( i_2 j_2 \):

Case II+B 1: \( a_1 \leq i_2 < b_1 \).

Case II+B 1 (a) \( b_1 \leq j_2 \leq i_1 \). Consider the cycle C differing from X in that the path from \( i_2 \) to \( j_2 \) in \( X \) is replaced in C by the edge \( i_2 j_2 \) and the path from \( i_1 \) to \( a_1 \) in \( X \) is replaced by the edge \( i_1 a_1 \). (See Figure 5.10.) Then cycles C and X have disconnected intersection, and by the proof of the intersection lemma,
Figure 5.10
G mimics MIN.1 or MIN.2.

Case II+B 1 (b): \( i_1 < j_2 \). Consider the cycle C following P from \( a_1 \) to \( i_2 \), then edge \( i_2 j_2 \), then P to \( b_2 \), then \( b_2 b_1 \), then P from \( b_1 \) to \( i_1 \), finally \( i_1 a_1 \). (See Figure 5.11.) Then cycles C and X have disconnected intersection, and by the intersection lemma, G mimics MIN.1 or MIN.2.

Case II+B 2: \( b_1 \leq i_2 < a_2 \). Since \( i_2 \in X \cap Y \), assume that \( j_2 \in Y \setminus X \), viz. \( a_2 < j_2 \). Then G mimics the triangle, MIN.1. (See Figure 5.12.)

In case II+C we make the following subcases based on \( i_2 j_2 \):

- **Case II+C 1:** \( a_1 \leq i_2 < j_1 \). (Thus \( b_1 \leq j_2 \).) Again G mimics the triangle. (See Figure 5.13.)

- **Case II+C 2:** \( j_1 \leq i_2 < b_1 \). Again, \( b_1 \leq j_2 \).

  **Case II+C 2 (a):** \( b_1 \leq j_2 \leq i_1 \). Consider the cycle C following P from \( j_1 \) to \( i_2 \), then edge \( i_2 j_2 \), then P to \( i_1 \), finally \( i_1 j_1 \). (See Figure 5.14.) Then cycles C and X have disconnected intersection, and by the intersection lemma, G mimics MIN.1 or MIN.2.
Figure 5.11

Figure 5.12

Figure 5.13

Figure 5.14
Case II+C 2 (b) \( i_1 < j_2 \). Consider the cycle \( C \) following \( P \) from \( j_1 \) to \( i_2 \), then edge \( i_2 j_2 \), then \( P \) to \( b_2 \), then \( b_2 b_1 \), then \( P \) from \( b_1 \) to \( i_1 \), finally \( i_1 j_1 \).
(See Figure 5.15). Then cycles \( C \) and \( X \) have disconnected intersection, and by the intersection lemma, \( G \) mimics \( \text{MIN.1} \) or \( \text{MIN.2} \).

Case II+C 3: \( b_1 < i_2 < a_2 \). (Thus \( a_2 < j_2 \)).
We have two cases:

(i) \( i_1 < a_2 \). Here \( G \) mimics the triangle. (See Figure 5.16)
(ii) \( i_1 = a_2 \). Here \( G \) mimics \( \text{MIN.22} \). (See Figure 5.17.)

Case II+C 4: \( i_2 = a_2 \). Here \( G \) mimics \( \text{MIN.2} \)
(See Figure 5.18)

We can use case II+C to attack case II+D. Suppose that \( G \) has an edge \( ij \) of type D. Then \( G \) can mimic a digraph \( H \) which falls under case C. (See Figure 5.19)
One walks 32 modulo paths on \( G \) by following \( P \) from \( a_2 \) to \( i \), then edge \( ij \). Similarly, one walks 43 by following edge \( b_2 b_1 \), then \( P \) from \( b_1 \) to \( a_2 \).
Suppose that $G$ also has a forward edge $kl$. Then usually, $G$ can mimic a digraph $H'$ derived from $H$ by adding a forward edge. Digraph $H'$ will fall under case II + C, and hence be versatile. An example is shown in Figure 5.20. Difficulties only arise if $b_1 \leq k$, $l \leq i$. In such a case, each of $k$ and $l$ lies on one of the paths on which we would walk edges 32 and 43 of $H$.

Suppose then that $b_1 \nless k$, $l \nless i$. By the symmetry of type D edges under reflection, we assume that $b_1 \nless k$, $l \nless a_2$. However this means that both $k$ and $l$ lie in $X \cap Y$, and this case was dealt with under II.1.

We have now shown that $i_1 j_1$ and $i_2 j_2$ may both be assumed to be back edges. A simple induction on the number of forward edges in $G$ shows that we may assume that every extraskeletal edge of $G$ is a back edge.

For economy of cases in the rest of this chapter, we will use the following case divisions:

1. Every extraskeletal edge of $G$ is a back edge of case E.

2. Graph $G$ has a back edge of case A and (up to reversal) every back edge of $G$ is a back edge of either case A or E.

3. Graph $G$ has a back edge of case B and (up to
Figure 5.20
reversal) every back edge of \( G \) is a back edge of either case \( B, A, \) or \( E. \)

(4) Graph \( G \) has a back edge of case \( C \) and (up to reversal) every back edge of \( G \) is a back edge of either case \( C, B, A \) or \( E. \)

(5) Graph \( G \) has a back edge of case \( D. \)

**Case (1):** Every extraskeletal edge of \( G \) is a back edge of case \( E. \)

If every edge of \( G \) is of type \( E \), then without loss of generality \( i_1 j_1 \) and \( i_2 j_2 \) form an \( M \) where \( a_1 < b_1 \leq j_1 < j_2 \leq i_1 < i_2 \leq a_2 < b_2. \) Here \( G \) mimics MIN.15. (See Figure 5.21)

**Case (2):** Graph \( G \) has a back edge of case \( A \) and (up to reversal) every back edge of \( G \) is a back edge of either case \( A \) or \( E. \)

Note that an edge of type \( A \) can never form an \( M \) with an edge of type \( E. \) Thus if \( G \) has an edge of type \( E, \) it will have two type \( E \) edges forming an \( M \) as in the previous case. Therefore we may assume in this case that \( G \) has only type \( A \) back edges.

If \( m = 2, \) then without loss of generality (invoking
the M lemma, and renaming if necessary: \( a_1 \approx j_1 \prec j_2 \prec i_1 \prec i_2 \prec b_1 \).

If \( a_1 = j_1 \), then a reduction of \( G \) is mimicked by MAX.7. (See Figure 5.22.)

If \( b_1 \) is the successor of \( i_2 \) on \( P \), then a reduction of \( G \) is mimicked by MAX.15. (See Figure 5.23.)

If \( a_1 \prec j_1 \) and there is a vertex of \( G \) between \( i_2 \) and \( b_1 \), then \( G \) mimics MIN.18. (See Figure 5.24.)

For the remainder of Case (2), assume that \( m > 2 \).

Suppose that \( G \) has edges \( i_1 j_1 \), a type A edge, and \( i_2 j_2 \), a type A edge after reversal, i.e. \( a_1 \approx j_1 \prec i_1 \prec b_1 \) and \( a_2 \prec j_2 \prec i_2 \prec b_2 \). Then without loss of generality, \( G \) also has edges \( i_3 j_3 \) and \( i_4 j_4 \) where \( a_1 \approx j_3 \prec j_1 \prec i_3 \prec i_1 \prec b_1 \approx a_2 \prec j_4 \prec j_2 \prec i_4 \prec i_2 \prec b_2 \). Here \( G \) mimics MIN.19. (See Figure 5.25.)

We may thus assume that every extra-skeletal edge \( i_s j_s \) of \( G \) is a true type A edge, viz. \( a_1 \approx j_s \prec i_s \prec b_1 \).

We now introduce a "stripping" method of classification, that will serve us again in the next chapter. Since \( G \) has only type A edges, and these only in the first "half" of \( G \), we strip away other edges of \( G \), and use these type A edges for our classification:

Let \( G' \) be the graph obtained from \( G \) by removing the
Figure 5.22

Figure 5.23

Figure 5.24

Figure 5.25
edges $a_2a_1$ and $b_2b_1$. Consider the strongly connected components of $G'$ consisting of more than one vertex. At least one such component exists, since $G$ has back edges. If more than one such component exists, then without loss of generality $G$ has edges $i_1j_1, i_2j_2, i_3j_3, i_4j_4$ where $a_1 < j_1 < j_2 < i_1 < i_2 < j_3 < j_4 < i_3 < i_4 < b_1$, since each of these components contains an $M$. Here $G$ mimics MIN.20. (See Figure 5.26)

Thus, without loss of generality, we may speak of the strongly connected component $G''$ of $G'$ containing more than one vertex. Since $G''$ is a strongly connected digraph, we may invoke our previous classification results to say things about the structure of $G''$. This is our "stripping" method.

**Case- $G''$ is a graph of type (3) of the Classification Lemma (Lemma 3.3):**

Without loss of generality $G$ has three back edges $i_1j_1, i_2j_2, i_3j_3$, with $a_1 < j_1 < j_2 < i_1 < i_2 < j_3 < i_3 < b_1$. Here $G$ mimics MIN.21. (See Figure 5.27)

**Case- $G''$ is a graph of type (2) of the classification lemma:**

**Subcase- $G''$ has edges of type E only:** As we
have seen previously, $G''$, hence $G$, mimics MIN.15.

**Subcase- $G''$ has edges of type $A$ only:** Here without loss of generality (up to reversing the roles of $X$ and $Y$ and reversal) $G$ has edges $i_1j_1, i_2j_2, i_3j_3, i_4j_4$ with

$a_1 \leq j_1 \leq j_2 < j_3 < i_2 < i_3 < i_4 \leq j_4 \leq i_1 < i_4 < b_1$. Here $G$ mimics MIN.66. (See Figure 5.28)

**Subcase- $G''$ has an edge of type $B$:** Here without loss of generality (up to reversing the roles of $X$ and $Y$ and reversal) $G$ has edges $i_1j_1, i_2j_2, i_3j_3$ with

$a_1 \leq j_1 = j_3 < j_2 < i_3 < i_2 < b_1$. Here $G$ mimics MIN.23. (See Figure 5.29)

**Subcase- $G''$ has an edge of type $C$:** Here without loss of generality (up to reversing the roles of $X$ and $Y$ and reversal) $G$ has edges $i_1j_1, i_2j_2, i_3j_3$ with

$a_1 \leq j_1 < j_3 < j_2 < i_3 \leq i_1 < i_2 < b_1$. Here $G$ mimics MIN.24. (See Figure 5.30)

**Subcase- $G''$ has an edge of type $D$:** Here without loss of generality (up to reversing the roles of $X$ and $Y$) $G$ has edges $i_1j_1, i_2j_2, i_3j_3$ with

$a_1 \leq j_1 < j_3 < j_2 < i_1 < i_3 < i_2 < b_1$. Here $G$ mimics
Thus if $G''$ falls under type (2) of the classification lemma, $G$ mimics a digraph of MIN.

Case- $G''$ is a graph of type (1) of the classification lemma:

Here without loss of generality $G$ has edges $i_1j_1$, $i_2j_2$, $i_3j_3$ with $a_1 < j_3 < j_2 < j_1 < i_2 < i_1 < i_3 < b_1$. (The edge $i_3j_3$ is the back edge of the skeleton of $G''$. Since $i_3j_3$ can be assumed to be a useful edge, we pick $i_1j_1$, $i_2j_2$ to form an $M$ under $i_3j_3$. Thus either $j_3 = j_2$, or $i_1 = i_3$. We may assume that $i_1 < i_3$ without loss of generality, up to reversal of $G$, or the interchanging of cycles $X$ and $Y$.)

We form subcases:

Subcase- $a_1 < j_2$: Here $G$ mimics MIN.18. (See Figure 5.32)

Subcase- $j_2 = a_1 = i_3$, $m = 3$: In this case a reduction of $G$ can be mimicked by MAX.14. (See Figure 5.33)

Subcase- $m > 3$: Either repeating our stripping process on $G''$ will lead to a graph of type (3) of the
Figure 5.31

Figure 5.32

Figure 5.33
classification lemma, or a graph of type (2) of the classification lemma with a back edge, or a graph of type (1) of the classification lemma.

We may assume, without loss of generality that the first two cases do not occur. Assume without loss of generality that $a_1 = j_2 = j_3$. Otherwise $G$ mimics a digraph of MIN as already shown above. Without loss of generality, up to reversal of $G$, or the interchanging of cycles $X$ and $Y$, $G$ contains an edge $i_4 j_4$ with $j_4 = a_1$ and $i_1 < i_4 < i_3$. Thus $G$ mimics MIN. (See Figure 5.34)

This completes our examination of Case (2).

**Case (3):** Graph $G$ has a back edge of type $B$ and (up to reversal) every back edge of $G$ is a back edge of either type $B$, $A$, or $E$.

In analogy to the previous case, we first dismiss the cases where not every edge of $G$ is a type $B$ edge.

Suppose that $G$ has an edge $i_2 j_2$ of type $E$. If the edge of $G$ intersecting $i_2 j_2$ to form an $M$ is a type $E$ edge, then we are done, as in case (1). Thus without loss of generality we may assume that

$$a_1 = j_1 < b_1 < j_2 < i_1 < i_2 < a_2 < b_2.$$
Figure 5.34
If we now interchange the roles of edges \( i_1 j_1 \) and \( a_2 a_1 \), then with respect to the new skeleton for \( G \), \( i_2 j_2 \) is an edge of type C. (See Figure 5.35). We may thus delay discussion of this possibility until case (4).

From now on, let us assume that \( G \) has no edges of type E.

Suppose that \( G \) has a type A edge \( i_2 j_2 \). Then without loss of generality, \( i_2 j_2 \) forms an M with a type A edge \( i_3 j_3 \). We have two possibilities:

(i) \( a_1 = j_1 \leq j_2 \leq j_3 \leq i_2 \leq i_3 \leq b_1 \leq i_1 \leq a_2 \). Here \( G \) mimics MIN.26. (See Figure 5.36)

(ii) \( a_1 \leq j_2 < j_3 \leq i_2 < i_3 \leq b_1 \leq j_1 \leq a_2 \leq i_1 = b_2 \). Here \( i_1 j_1 \) is a reversed type B edge. Here \( G \) mimics MIN.27. (See Figure 5.37)

For the remainder of this case we assume that \( G \) has (up to reversal) only type B edges. For convenience, we rename edges here:

Let \( i_1 j_1, i_2 j_2, \ldots, i_r j_r \) be the (normal) type B edges of \( G \)

\[ j_1 = j_2 = \ldots = j_r = a_1 \leq b_1 \leq i_1 < i_2 < \ldots < i_r < a_2. \]

Let \( k_1 l_1, k_2 l_2, \ldots, k_s l_s \) be the (reversed) type B edges
edges of $G$

$b_1 < l_1 < l_2 < \ldots < l_s \leq a_2 < k_1 = k_2 = \ldots = k_3 = b_2$.

If $s = 0$ then for large enough $q$, a reduction of $G$ is mimicked by MAX.13. (See Figure 5.38)

If $s > 0$ and for some $t$ and $u$, $l_t \leq i_u$, then $G$ mimics MIN.28. (See Figure 5.39)

We may thus assume from now on that $r > s > 0$, and $l_t > i_u$ for all $t, u$ where $1 \leq u \leq r$, $i \leq t \leq s$.

Our remaining subcases are based on the values of $s$ and $r$.

Subcase $r < 3, s = 1$: Here a reduction of $G$ is mimicked by MAX.12. (See Figure 5.40)

Subcase $r = 2, s > 1$: Here $G$ mimics MIN.29.

(See Figure 5.41.)

Subcase $r > 2$: Here $G$ mimics MIN.30.

(See Figure 5.42.)

This concludes our examination of case (3).

It will prove economical to deal with case (5) before
case (4).

**Case (5):** Graph G has a back edge of case D.

Note that case (5) shows mirror symmetry; If G is reversed, then switching the roles of $a_1, a_2, b_1, b_2, i_1, j_1$ with $b_2, b_1, a_2, a_1, j_1, i_1$ respectively again gives us case (5). This symmetry allows us to reduce our number of cases.

$m = 2$:

We make the following case divisions based on $j_2$:

Case $D\alpha$: $j_2 < j_1$.

Case $D\beta$: $j_1 \leq j_2 < b_1$ ( $i_2 \leq i_1$ by symmetry ).

Case $D\gamma$: $b_1 \leq j_2 \leq a_2$ ( $i_2 \leq a_2$ ). In this case the edge $i_2j_2$ is useless, a contradiction. ( See Figure 5.43 )

Case $D\alpha$ is subdivided as follows depending on $i_2$:

Case $D\alpha 1$: $i_2 < j_1$. In this case the edge $i_2j_2$ is useless, a contradiction. ( See Figure 5.44 )

Case $D\alpha 2$: $j_1 \leq i_2 < i_1$. We have two possibilities:

(i) $i_2 \neq a_2$. Here G mimics MIN.31. ( See Figure 5.45. Here the greater of $a_2, i_2$ is labelled
Figure 5.43

Figure 5.44
(ii) \( i_2 = a_2 \). Here \( G \) mimics MIN.32. (See Figure 5.46)

Case Da3: \( i_1 < i_2 \). (Thus either \( i_1 < i_2 \), or \( i_2 < b_2 \). Note that if \( j_2 = a_1 \) and \( i_2 = b_2 \), then \( G \) is a one hump digraph. By symmetry, assume that \( i_2 < b_2 \).)

Here \( G \) mimics MIN.33. (See Figure 5.47.)

In Case Dp, we may assume that \( i_2 < i_1 \), otherwise interchanging the roles of \( i_1 j_1, i_2 j_2 \) gives Case Da. Case Dp is subdivided as follows depending on \( i_2 \):

Case Dp1: \( i_2 < b_1 \). In this case the edge \( i_2 j_2 \) is useless, a contradiction. (See Figure 5.48)

Case Dp2: \( b_1 < i_2 \). Here \( G \) mimics MIN.32. (See Figure 5.49)

This concludes the subcase when \( m = 2 \).

\( m > 2 \):

By the foregoing analysis we may assume that any extra-skeletal edge of \( G \) falls (up to reversal) under cases Da1, Dp1 or D7.
Case - Edges of type D\(\text{a}1\): Without loss of generality, invoking the M lemma, \(G\) has edges \(i_2 j_2, i_3 j_3\), with \(a_1 \leq j_2 < j_3 \leq i_2 < i_3 < j_1\). In this case, \(G\) mimics MIN.14. (See Figure 5.50)

Case - Edges of type D\(\text{a}1\): Without loss of generality, \(G\) has edges \(i_2 j_2, i_3 j_3\), with \(j_1 \leq j_2 < j_3 \leq i_2 < i_3 \leq b_1\). In this case, \(G\) mimics MIN.15. (See Figure 5.51)

Case - Edges of type D\(\text{f}\): Without loss of generality, \(G\) has edges \(i_2 j_2, i_3 j_3\), with \(b_1 \leq j_2 < j_3 \leq i_2 < i_3 < a_2\). In this case, \(G\) mimics MIN.15, as in case (1). (See Figure 5.52)

This concludes our examination of case (5).

Case (4): Graph \(G\) has a back edge of case C and (up to reversal) every back edge of \(G\) is a back edge of either case C, B, A or E.

\(m = 2\):

We form cases based on \(j_2\).

Case C\(\text{a}\): \(j_2 < j_1\). (\(j_2 = a_1\), or not \(a_2 < i_2 < b_2\))

Case C\(\beta\): \(j_1 \leq j_2 < b_1\). (not \(a_2 < i_2 < b_2\))
Figure 5.50

Figure 5.51

Figure 5.52
Case C7: $b_1 \leq j_2 \leq i_1$.

Case C6: $i_1 < j_2 \leq a_2$.

Case C6: $a_2 < j_2$. In this case the edge $i_2j_2$ is useless, a contradiction. (See Figure 5.53)

We form subcases of case Cα based on $i_2$.

Case Call: $i_2 < j_1$. In this case the edge $i_2j_2$ is useless, a contradiction. (See Figure 5.54)

Case Ca2: $j_1 \leq i_2 < b_1$. Here make three further divisions:

(i) $j_2 \not= a_1$. In this case, $G$ mimics MIN.16. (See Figure 5.55)

(ii) $j_2 = a_1$, $i_1 < a_2$. In this case, $G$ mimics MIN.34. (See Figure 5.56)

(iii) $i_1 = a_2$, $j_2 = a_1$. In this case, a reduction of $G$ is mimicked by MAX.11. (See Figure 5.57)

Case Ca3: $b_1 \leq i_2 \leq i_1$ Here make two further divisions:

(i) $i_2 < i_1$. In this case, $G$ mimics MIN.33. (See Figure 5.58)
(ii) \( i_2 = i_1 \). In this case, a reduction of \( G \) is mimicked by MAX.10. (See Figure 5.59)

Case Ca4: \( i_1 < i_2 \leq a_2 \). Here make two further divisions:

(i) \( a_1 = j_2 \). In this case, \( G \) mimics MIN.36. (See Figure 5.60)

(ii) \( a_1 < j_2 \). In this case, \( G \) mimics MIN.37. (See Figure 5.61)

Case Ca5: \( a_2 < i_2 < b_2 \). Here \( a_1 = j_2 \), or else \( G \) has a type D edge. Interchanging the roles of \( a_2 a_1 \) and \( i_2 j_2 \) gives case Ca4 or Ca3 (ii) when \( i_1 = a_2 \). (See Figure 5.62)

Case Ca6: \( i_2 = b_2 \). In this case, a reduction of \( G \) is mimicked by MAX.9. (See Figure 5.63)

Case Cb is subdivided as follows depending on \( i_2 \):

Case Cb1: \( i_2 < b_1 \). In this case the edge \( i_2 j_2 \) is useless, a contradiction. (See Figure 5.64)
Figure 5.59

Figure 5.60

Figure 5.61
Figure 5.62

Figure 5.63

Figure 5.64
Case C\(p2\): \(b_1 \leq i_2 \leq a_2\). Here we assume \(j_1 = j_2\). Otherwise, reversing the roles of \(i_1 j_1\) and \(i_2 j_2\) gives rise to case Ca3 or Ca4. In this case, a reduction of \(G\) is mimicked by MAX.2. (See Figure 5.65)

Case C\(p3\): \(a_2 < i_2\). (Thus assume \(i_2 = b_2\), since \(G\) has no edges of type D.) Here make two further divisions:

(i) \(j_1 = j_2\). In this case, a reduction of \(G\) is mimicked by MAX.4. (See Figure 5.66)

(ii) \(j_1 < j_2\). In this case, \(G\) mimics MIN.38. (See Figure 5.67)

Case C7 is subdivided as follows depending on \(i_2\):

Case C\(71\): \(i_2 \leq i_1\). In this case the edge \(i_2 j_2\) is useless, a contradiction. (See Figure 5.68)

Case C\(72\): \(i_1 < i_2 < b_2\). In this case, if \(i_1 < a_2\) then \(G\) mimics MIN.55. (See Figure 5.69. Here 4 labels the lesser of \(a_2, i_2\) w.r.t. P.) If \(i_1 = a_2\), then \(G\) mimics MIN.77. (See Figure 5.70.)

Case C\(73\): \(i_2 = b_2\). Here interchanging names
Figure 5.68

Figure 5.69

Figure 5.70
of $i_2 j_2$ and $b_2 b_1$ puts us in case $C\rho 3(ii)$ (See Figure 5.71.)

Case $C\rho$ is subdivided as follows depending on $i_2$:

Case $C\rho 1$: $i_2 < a_2$. In this case the edge $i_2 j_2$ is useless, a contradiction. (See Figure 5.72.)

Case $C\rho 2$: $a_2 < i_2 < b_2$. If $G$ has no vertex between $i_1$ and $j_2$ on $P$, then a reduction of $G$ is mimicked by MAX.8. (See Figure 5.73.)

However, if there is a vertex $x$ of $G$ between $i_1$ and $j_2$, then $G$ mimics MIN.40. (See Figure 5.74.)

Case $C\rho 3$: $i_2 = b_2$. In this case, a reduction of $G$ is mimicked by MAX.5. (See Figure 5.75.)

This concludes the subcase when $m = 2$. In most cases we showed that $G$ mimicked a graph of MIN. In cases $C\alpha 1$, $C\beta 1$, $C\gamma 1$, $C\delta 1$, $C\epsilon$ the edge $i_2 j_2$ was useless. The other cases were:

- $Ca3$ (ii) with $a_1 < j_2$, $C\beta 2$
- $Ca6$, $Ca3$ (i)
- $C\delta 2$, $C\delta 3$
- $Ca2$ (iii)
Ca3 (ii) with $a_1 = j_2$

We have grouped these cases according to similarities which are evident in the figures given for these cases. To aid the memory (not wishing the reader to have to recall what case Ca3 (ii) edges look like and so forth), we reflect these similarities in a renaming of cases; refer to edges falling under cases Ca3 (iii), $\psi 2$ as type C1 (a), C1 (b) edges respectively, Ca6, $\psi 3$ (i) as type C2 (a), C2 (b) edges respectively, Cc2, Cc3 as type C3 (a), C3 (b) edges respectively, Ca2 (iii) as type C4 edges, Ca3 (ii) as type C5 edges.

Since these terms will be used in further breakdowns, the reader is advised to review the named cases so as to have at his finger tips what the edges of these various types look like.

**Useless Edges:**

Next we consider the cases when $m \geq 3$, and $i_2 j_2$ satisfies the conditions given in one of cases Ca1, $\psi 1$, C71, C51, Cc.
**Edge** $i_2 j_2$ **falls under Ca1:** Without loss of generality, edge $i_2 j_2$ forms an M with some edge $i_3 j_3$ where $j_3 < b_1$. This gives three subcases:

Subcase- $i_3 j_3$ is an edge of type C1(a): Without loss of generality, $a_1 < j_2 < j_3 < i_2 < j_1 < b_1 < i_3 = i_1$. In this case, $G$ mimics MIN.39. (See Figure 5.76)

Subcase- $i_3 j_3$ is an edge of type C2(a): Without loss of generality, $a_1 < j_2 < j_3 < i_2 < j_1 < i_3 = b_2$. In this case, $G$ mimics MIN.41. (See Figure 5.77)

Subcase- $i_3 j_3$ is an edge of type Ca1: Without loss of generality, $a_1 < j_2 < j_3 < i_2 < i_3 < j_1$. We make a further subdivision:

(i) $a_1 < j_2$. In this case, $G$ mimics MIN.18. (See Figure 5.78)

(ii) $a_1 = j_2$, $i_1 < a_2$. In this case, $G$ mimics MIN.27. (See Figure 5.79)

(iii) $a_1 = j_2$, $i_1 = a_2$ and $m = 3$. In this case, a reduction of $G$ is mimicked by MAX.7. (See Figure 5.80)
What happens if $i_2 j_2$ and $i_3 j_3$ are Cal edges forming an M, $a_1 = j_2$, $i_1 = a_2$ and $m > 3$? We shall see in a moment that if $m \geq 3$ and G has an edge of type C₀₁, C₂₁, C₃₁, or C₄, then G mimics a graph of MIN. We shall thus assume here that G has no such edges. We thus assume that edge $i_4 j_4$ falls under one of cases Cal, C₁₁, C₂, C₃, C₄, C₅.

Since $i_1 = a_2$, it follows that G has no edges of types C₁(b), C₃ or C₅. It also follows from previous discussion concerning the edges of type Cal that we may assume that the only edges forming M's with Cal edges are themselves Cal edges. Only a few cases are left:

- G has an edge of type C₁(a): Here G mimics MIN.27. (See Figure 5.81.)

- G has an edge of type C₂: Here G mimics MIN.27. (See Figure 5.82.)

- G has an edge of type C₄: Here G mimics MIN.14. (See Figure 5.83.)

Every extraskeletal edge of G other than $i_1 j_1$ is a Cal edge: By our previous breakdown of the case when every extraskeletal edge of G is a type A edge, we may assume that G has an edge $i_4 j_4$ where $a_1 = j_4 = j_2 < j_3 < i_2 < i_3 < i_4 < j_1 < b_1 < i_1 = a_2$. In
Figure 5.81

Figure 5.82

Figure 5.83
this case, G mimics MIN.42. (See Figure 5.84.)

This concludes the case where G has an edge of type Cα.

**Edge i₂j₂ falls under Cα:** Without loss of generality, edge i₂j₂ forms an M with some edge i₃j₃ where j₁ < j₃ < b₁ and/or j₁ ≤ i₃ < b₁. The candidate subcases are:

- **Subcase- i₃j₃ is an edge of type Cβ1:** Without loss of generality, j₁ ≤ j₂ < j₃ ≤ i₂ < i₃ < b₁. In this case, G mimics MIN.15. (See Figure 5.85)

- **Subcase- i₃j₃ is an edge of type C1(a):** Without loss of generality, j₁ ≤ j₂ < j₃ ≤ i₂ < i₃ < b₁. In this case, G mimics MIN.16. (See Figure 5.86)

- **Subcase- i₃j₃ is an edge of type C4:** Without loss of generality, j₁ < j₂ as separate. Without loss of generality, j₁ ≤ j₂ < j₃ ≤ i₂ < i₃ = i₁. In this case, G mimics MIN.16. (See Figure 5.86)

- **Subcase- i₃j₃ is an edge of type C4:** Without loss of generality, a₁ = j₃ < j₁ ≤ j₂ ≤ i₃ < i₂. In this case, G mimics MIN.43. (See Figure 5.87)
Figure 5.84

Figure 5.85

Figure 5.86
Edge $i_2 j_2$ falls under C71: Without loss of generality, edge $i_2 j_2$ forms an M with some edge $i_3 j_3$ where $b_1 < j_3 < i_1$ and/or $b_1 < i_3 < i_1$. The candidate subcases are:

Subcase- $i_3 j_3$ is an edge of type C71: In this case $G$ has two type E edges forming an M, and we are done as in case (1).

Subcase- $i_3 j_3$ is an edge of type C1(b). Without loss of generality, $j_1 = j_3 < b_1 < j_2 < i_3 < i_2 < i_1$. In this case, $G$ mimics MIN.55. (See Figure 5.88)

Edge $i_2 j_2$ falls under C51: Without loss of generality, edge $i_2 j_2$ forms an M with some edge $i_3 j_3$ where $i_1 < j_3 < a_2$ and/or $i_1 < i_3 < a_2$. The candidate subcases are:

Subcase- $i_3 j_3$ is an edge of type C51: In this case $G$ has two type E edges forming an M, and we are done as in case (1).

Subcase- $i_3 j_3$ is an edge of type C1(b). Without loss of generality, $j_1 = j_3 < b_1 < i_1 < j_2 < i_3 < i_2 < a_2$. In this case, $G$ mimics MIN.15. (See Figure 5.89)

Subcase- $i_3 j_3$ is an edge of type C3. Actually
Figure 5.87

Figure 5.88

Figure 5.89
in C3(a), there is no vertex between \( i_1 \) and \( j_2 \) ( \( = j_3 \) here) so only C3(b) is possible. Without loss of
generality, \( b_1 \leq i_1 < j_2 < j_3 \leq i_2 \leq a_2 < i_3 \). In this
case, G mimics MIN.41. (See Figure 5.90)

**Edge \( i_2 j_2 \) falls under C\( 5 \):** Without loss of
generality, edge \( i_2 j_2 \) forms an M with some edge \( i_3 j_3 \)
where \( a_2 < j_3 \) and/or \( a_2 < i_3 \). The candidate subcases
are:

- **Subcase-** \( i_3 j_3 \) is an edge of type C\( 5 \). Without
  loss of generality, \( a_2 < j_3 < j_2 < i_3 < i_2 \). In this case,
  G mimics MIN.56.
  (See Figure 5.91)

- **Subcase-** \( i_3 j_3 \) is an edge of type C3(a). Without
  loss of generality, \( i_1 < j_3 \leq a_2 < j_2 < i_3 < i_2 \). In this
  case, G mimics MIN.34. (See Figure 5.92)

This completes the cases where \( m \geq 3 \), and \( i_2 j_2 \)
satisfies the conditions given in one of cases C\( 0 \), C\( 1 \), 
C\( 1 \), C\( 1 \), C\( 5 \), C\( 5 \).

**No Useless Edges**

From now on we assume that G does not have edges of
types C\( 0 \), C\( 1 \), C\( 1 \), C\( 5 \), C\( 5 \). Thus every extraskeletal
edge of G is of one of types C1, C2, C3, C4, C5.
m = 3:
Another level of cases will prove useful. We create five new cases:

(5.1) G contains a C5 edge \(i_2j_2\).

(5.2) G contains a C4 edge \(i_2j_2\), but no C5 edge.

(5.3) G contains a C3 edge \(i_2j_2\), but no C4 or C5 edges.

(5.4) G contains a C2 edge \(i_2j_2\), but no C3, C4 or C5 edges.

(5.5) G contains only C1 edges.

Case (5.1) is subdivided according to \(i_3j_3\):

Note that \(i_3j_3\) cannot be a C4 edge, since \(i_1 \neq a_2\). Also \(i_3j_3\) cannot be a C5 edge, since then \(i_2j_2\) and \(i_3j_3\) would be equal.

**Subcase C5 + C3(a):** Edge \(i_3j_3\) is a C3(a) edge.

In this case, G mimics MIN.44. (See Figure 5.93)

**Subcase C5 + C3(b):** Edge \(i_3j_3\) is a C3(b) edge.

We have two possibilities:

(i) There is a vertex \(x\) of P between \(i_1\) and \(j_3\). In this case, G mimics MIN.45. (See Figure 5.94)

(ii) There is no vertex of P between \(i_1\) and \(j_3\). In this case, a reduction of G is mimicked by MAX.6. (See Figure 5.95)
Figure 5.93

Figure 5.94
Subcase C5 + C2: Edge $i_3j_3$ is a C2 edge. In this case, $G$ mimics MIN.28. (See Figure 5.96)

Subcase C5 + C1: Edge $i_3j_3$ is a C1 edge. Then $i_3j_3$ may be assumed to be a C1(a) edge, or else by considering edges $i_3j_3$ and $i_2j_2$, $G$ falls under one of case Ca3(i), or case Ca4.
In this case, $G$ mimics MIN.46. (See Figure 5.97)

Case (5.2) is subdivided according to $i_3j_3$: Note that $i_3j_3$ cannot be a C3 edge, since $i_1 = a_2$.

Subcase C4 + C4: Edge $i_3j_3$ is a C4 edge. In this case, $G$ mimics MIN.5. (See Figure 5.98)

Subcase C4 + C2: Edge $i_3j_3$ is a C2 edge. In this case, $G$ mimics MIN.47. (See Figure 5.99)

Subcase C4 + C1(a): Edge $i_3j_3$ is a C1(a) edge. In case C1(a) (nee Ca3) we assumed without loss of generality that $j_2 < j_1$. With an additional C4 edge and relabelling this assumption gives way to two possibilities:

(i) $j_1 < i_2 < j_3$. In this case, $G$ mimics
MIN.39.
(See Figure 5.100)

(ii) \( j_3 \leq i_2 \). In this case, \( G \) mimics MIN.7.
(See Figure 5.101)

**Subcase \( C_4 + C_1(b) \):** Edge \( i_3j_3 \) is a \( C_1(b) \) edge.
In this case, \( G \) mimics MIN.7. (See Figure 5.102)

Case (5.3) is subdivided according to \( i_3j_3 \):

**Subcase \( C_3 + C_3 \):** Edge \( i_3j_3 \) is a \( C_3 \) edge. We have two possibilities:

(i) At least one of \( i_2j_2, i_3j_3 \) is a \( C_3(a) \) edge. In this case, \( G \) mimics MIN.44. (See Figure 5.103)

(ii) Both \( i_2j_2, i_3j_3 \) are \( C_3(b) \) edges. In this case a reduction of \( G \) is mimicked by MAX.5. (See Figure 5.104)

**Subcase \( C_3 + C_2 \):** Edge \( i_3j_3 \) is a \( C_2 \) edge. We have three possibilities:

(i) Edge \( i_3j_3 \) is a \( C_2(a) \) edge. In this case \( G \) mimics MIN.48. (See Figure 5.105)
(i) Edge $i_2j_2$ is a C3(a) edge. In this case $G$ mimics MIN.49. (See Figure 5.106)

(iii) Edge $i_2j_2$ is a C3(b) edge and edge $i_3j_3$ is a C2(b) edge. In this case a reduction of $G$ is mimicked by MAX.4. (See Figure 5.107)

**Subcase C3 + C1(a):** Edge $i_3j_3$ is a C1(a) edge. In this case, $G$ mimics MIN.50. (See Figure 5.108)

**Subcase C3 + C1(b):** Edge $i_3j_3$ is a C1(b) edge. In this case, $G$ mimics MIN.51. (See Figure 5.109)

Case (5.4) is subdivided according to $i_3j_3$:

**Subcase C2 + C2:** Edge $i_3j_3$ is a C2 edge. In this case, $G$ mimics MIN.52. (See Figure 5.110)

**Subcase C2 + C1:** Edge $i_3j_3$ is a C1 edge. We have three possibilities:

1. Edge $i_3j_3$ is a C1(a) edge. Without loss of generality, $j_2 \leq j_3$, otherwise we get case C93(ii). In this case $G$ mimics MIN.53. (See Figure 5.111)
Figure 5.109

Figure 5.110

Figure 5.111
(i) Edge $i_3 j_3$ is a Cl(b) edge, but edge $i_2 j_2$ is a C2(a) edge. In this case $G$ mimics MIN.81. (See Figure 5.112)

(iii) Edge $i_2 j_2$ is a C2(b) edge and edge $i_3 j_3$ is a Cl(b) edge. In this case a reduction of $G$ is mimicked by MAX.3. (See Figure 5.113)

Case (5.5) is subdivided. Every edge here will be a type Cl edge. We may assume that these edges do not cross, for otherwise suppose that $j_2 < j_3 < i_2 < i_3$. Then $G$ mimics MIN.33. (See Figure 5.114)

(i) Edges $i_2 j_2$, $i_3 j_3$ are Cl(a) edges. In this case $G$ mimics MIN.54. (See Figure 5.115)

(ii) Edge $i_2 j_2$ is a Cl(a) edge and edge $i_3 j_3$ is a Cl(b) edge. We make a distinction:

\[ j_3 < j_2 = j_1 < i_2 < i_3. \]

In this case $G$ mimics MIN.37. (See Figure 5.116)

\[ j_3 = j_2 < j_1 < i_1 = i_2 < i_3. \]
case \( G \) mimics MIN.37. (See Figure 5.117)

(iii) Edges \( i_3 j_3 \) and \( i_2 j_2 \) are \( C_1(b) \) edges, viz \( j_1 = j_2 = j_3 \). In this case a reduction of \( G \) is mimicked by MAX.2.
(See Figure 5.118)

This completes the case where \( m = 3 \).

\( m > 3 \):

In view of the previous section, we may assume that \( G \setminus i_4 j_4 \) falls under one of cases \( C_5 + C_3(b) \) (ii), \( C_3(b) + C_3(b) \), \( C_3(b) + C_2(b) \), \( C_2(b) + C_1(b) \) or \( C_1(b) + C_1(b) \). We consider these cases one by one:

\( G \setminus i_4 j_4 \) falls under \( C_5 + C_3(b) \) (ii):

We may also assume that \( G \setminus i_3 j_3 \) falls under case \( C_5 + C_3(b) \). We consider these cases one by one:

\( G \setminus i_3 j_3 \) falls under \( C_5 + C_3(b) \) (i):

From now on assume that \( G \) has no edge of type \( C_5 \).

\( G \setminus i_4 j_4 \) falls under \( C_3(b) + C_3(b) \):

We may assume one of two alternatives:

\( G \setminus i_3 j_3 \) falls under \( C_3(b) + C_3(b) \). In this case \( G \) mimics MIN.57. (See Figure 5.119)
Figure 5.117

Figure 5.118
G \ i_3 j_3 falls under C3(b) + C2(b). In this case G mimics MIN.58. (See Figure 5.120.)

We assume from now on that G contains no type C3 edges.

G \ i_4 j_4 falls under C2(b) + C1(b):

Here, if i_4 j_4 falls into types C2 or C1(a), then G \ i_3 j_3 falls under a case previously disposed of. We may therefore assume that i_4 j_4 is an edge of type C1(b).

Without loss of generality, say that

j_1 = j_3 = j_4 < i_4 < i_3 < i_1. In this case G mimics MIN.59. (See Figure 5.121.)

G \ i_4 j_4 falls under C1(b) + C1(b):

Assume without loss of generality that i_4 j_4 is a C1(b) edge, and

j_4 = j_3 = j_2 = j_1 < i_4 < i_3 < i_2 < i_1. In this case G mimics MIN.60. (See Figure 5.122.)

This concludes our classification, and the proof of this chapter's theorem.
Chapter 6: One Hump Digraphs

**Definition:** Let $G$ be a one hump digraph. Then by the definition of one hump digraphs, we can write $\text{vert}(G) = \text{vert}(P)$ where $P$ is a directed Hamiltonian path in $G$. Also $G$ has at least one additional edge $c_1d_1$ where $c_1$ is the terminal vertex of $P$, $d_1$ the initial vertex of $P$. Call $P \cup c_1d_1$ the **skeleton** of $G$. Any edge of $G$ which is not in the skeleton is called an **extra-skeletal** edge of $G$.

**Lemma 6.1:** Let $G$ be a one hump digraph with no useless edges, not mimicking a digraph in $\text{MIN}$. We may choose a skeleton $P \cup c_1d_1$ for $G$ so that with respect to $P$, every extra-skeletal edge of $G$ is a back edge.

**Proof:** Let $Q$ be the subgraph of $G \setminus c_1d_1$ induced by $P$. Let $J$ be the strongly connected component of $Q$ containing $c_1$. Let us suppose that we have chosen $P$ to make $|J|$ as large as possible. We will show that in this case, every extra-skeletal edge of $G$ is a back edge with respect to $P$.

Suppose that $G$ has an edge $kl$ which is a forward edge with respect to $P$.

**Case 1:** We have $k, l \in \text{vert}(J)$.

Then there is a cycle $C$ in $J$ containing the edge $kl$. But the intersection of cycles $P \cup c_1d_1$ and $C$ is not connected, so $G$ mimics $\text{MIN}.1$ or $\text{MIN}.2$ by the Intersection
Lemma. (See Figure 6.1)

**Case 2:** We have $k \in \text{vert}(Q \setminus J)$, $l \in \text{vert}(J)$.
Then we find a new skeleton for $Q$ replacing $c_1d_1$ by the edge $km$, where $m$ is the successor of $k$ in $P$, and replacing $P$ by the path $P'$ in $P \cup c_1d_1$ from $m$ to $k$. Then the strongly connected component of $G \setminus km$ contains $l$, hence all of $J \cup \{k\}$, contradicting our choice of $P$. (See Figure 6.2)

**Case 3:** We have $k, l \in \text{vert}(Q \setminus J)$.
By the last two cases, we may suppose that every forward edge of $G$ with respect to $P$ has both ends in $Q \setminus J$.
However, by the maximality of $J$, we may assume that $J$ contains more than one vertex, for if we pick $P$ to end at $k$, then $J$ contains $\{k, l\}$. Thus $J$ has a back edge, and contains some $M$, since $G$ has no useless edges. Then $G$ mimics $\text{MIN.18}$. (See Figure 6.3.)

We therefore conclude that every edge of $G$ is a back edge with respect to $P$.

**Proof of Theorem 3.10:** The proof of this theorem, which takes up the body of this chapter, proceeds by classifying the one hump digraphs. As usual, assume that
G is without useless edges. We can assume that every extra-skeletal edge of G is a back edge.

It is in this chapter that the "stripping" procedure introduced in the last chapter comes into its own. "Stripping" away edge $c_1d_1$ from G, we form cases based on the strongly connected components of $G \setminus c_1d_1$. A strongly connected component of $G \setminus c_1d_1$ which contains more than one vertex is called a bubble. We can assume that G has at least one bubble, since G contains an M. Our first level of subdivisions in the one hump case depends on the number of bubbles in G.

G has three or more bubbles
G has two bubbles
or G has only one bubble.

We look at these possibilities one by one:

Subcase: G contains three or more bubbles:

Without loss of generality, we may assume that G has back edges $i_1j_1$, $i_2j_2$, $i_3j_3$, $i_4j_4$, $i_5j_5$, $i_6j_6$ where $j_1 < j_2 < i_1 < i_2 < j_3 < j_4 < i_3 < i_4 < j_5 < j_6 < i_5 < i_6$. This is because each bubble of G must contain an M, as in Case 3 of Lemma 6.1. In this case G mimics MIN.61. (See Figure 6.4.)

Subcase: G contains two bubbles. Refer to the bubbles of G as $C_1$ and $C_2$ respectively. Without loss of
Figure 6.4
generality we may assume that each of $C_1$, $C_2$ is a one hump, two hump or three hump digraph, by the classification lemma. For the sake of definiteness, say that the vertices of $C_1$ precede those of $C_2$ on $P$. However, note that if we so desire, we can reverse the order of $C_1$ and $C_2$ on $P$ by putting $G$ into normal form in a different way: Simply rotate the skeleton of $G$. (See Figure 6.5.) Thus $C_1$ and $C_2$ are interchangeable. We now form cases based on $C_1$ and $C_2$.

**Subcase 1**: One of $C_1$, $C_2$ is a three hump digraph. Assume without loss of generality that $C_1$ is a three hump digraph. In any case, $C_2$ contains an $M$. Therefore $G$ has back edges $i_1 j_1$, $i_2 j_2$, $i_3 j_3$, $i_4 j_4$, $i_5 j_5$ where $j_1 < j_2 < i_1 < j_3 < i_2 < j_4 < i_3 < j_5 < i_4 < i_5$; In this case $G$ mimics MIN.62. (See Figure 6.6.)

**Subcase 2**: Both of $C_1$, $C_2$ are two hump digraphs.

**Subcase 3**: One of $C_1$, $C_2$ is a two hump digraph, and the other is a one hump digraph.

**Subcase 4**: Both of $C_1$, $C_2$ are one hump digraphs.

The point of our "stripping" classification is to
Figure 6.5

Figure 6.6
make use here of work done in the previous chapters. Consider the situation in subcase $p$. Here $G$ has at least four extra-skeletal edges $a_2a_1', b_2b_1', a_2'a_1', b_2'b_1'$ where the skeleton of $C_1$ consists of the edges $a_2a_1, b_2b_1$ and the path in $P$ from $a_1$ to $b_2$, and the skeleton of $C_2$ consists of the edges $a_2'a_1', b_2'b_1'$ and the path in $P$ from $a_1'$ to $b_2'$. Any further extra-skeletal edges of $G$ appear as extra-skeletal edges in $C_1$ or $C_2$, and this leads to the following subdivision of subcase $p$:

**Subcase $p.1$:** $G$ has exactly four extra-skeletal edges

Recall that the skeleton of $G$ is a cycle. This cycle gives a circular order to the vertices of $G$. We have two possibilities:

(i) No vertex of $C_1$ is a predecessor of a vertex of $C_2$ in the circular order, and no vertex of $C_2$ is a predecessor of a vertex of $C_1$ in the circular order. In this case, without loss of generality up to rotation of the skeleton, we may assume that $G$ has a vertex $x$ where $b_2 < x < a_1'$, and that $d_1 < a_1$. Then $G$ mimics MIN. (See Figure 6.7.)

(ii) Case (i) does not occur. Thus if $G$ has a vertex $x$ where $b_2 < x < a_1'$ then $d_1 = a_1$ and $b_2 = c_1$. In this case, a reduction of $G$ is mimicked by MAX. (See
Figure 6.7
**Subcase 6.2:** G has a fifth extra-skeletal edge.

Without loss of generality, (up to rotation and reversal), \( C_1 \) has one or more extra-skeletal edges falling into the categories of the previous chapter. This gives five possibilities:

(i) \( C_1 \) has an edge of type B of the previous chapter. In this case \( G \) mimics MIN.64. (See Figure 6.9.)

(ii) \( C_1 \) has an edge of type C of the previous chapter. In this case \( G \) mimics MIN.65. (See Figure 6.10.)

(iii) \( C_1 \) has an edge of type D of the previous chapter. In this case \( G \) mimics MIN.65. (See Figure 6.11.)

(iv) \( C_1 \) has no edges of types B, C, D, however, \( C_1 \) does have an edge of type E of the previous chapter. We may thus assume that \( C_1 \) has two type E edges forming an M, and, as in the previous chapter, \( C_1 \) mimics MIN.15. (See Figure 6.12.)
(v) \( C_1 \) has no edges of types B, C, D, however, \( C_1 \) does have an edge of type A of the previous chapter. We may thus assume that \( C_1 \) has two type A edges forming an \( M \). In this case, \( G \) mimics MIN. (See Figure 6.13.)

This concludes our consideration of subcase \( \beta \). We next consider subcase \( \gamma \). Without loss of generality, assume that \( C_1 \) is a one hump digraph, and \( C_2 \) is a two hump digraph. Repeatedly apply our stripping procedure to \( C_1 \). Eventually we arrive at a digraph \( C_1' \) which is a two hump or three hump digraph. If \( C_1' \) is a three hump digraph, then \( G \) mimics a digraph of MIN as in subcase \( \alpha \). Therefore assume without loss of generality that \( C_1' \) is a two hump digraph.

By our examination of subcase \( \beta \), we may assume that neither \( C_1' \) nor \( C_2 \) has extra-skeletal edges. Then without loss of generality, using rotations and reflections, the structure of \( G \) is as follows:

\[ G \] has extra-skeletal edges

\[ a_2 a_1', b_2 b_1', a_2 a_1', b_2 b_1', i_1 j_1, i_2 j_2, \ldots, i_s j_s \]

with \( a_1' < b_1 < a_2 < b_2 < a_1 < b_1 < a_2 < b_2 \)

\( j_s < j_s < \ldots < j_1 < a_1 < b_2 < i_1 < \ldots \)

\( \leq i_{s-1} \leq i_s < a_1' \).

and not both \( j_1 = a_1 \) and \( i_1 = b_2 \) (since otherwise edge
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\[ i_1 j_1 \text{ would be useless, not properly containing the } M \]
\[ \text{formed by } a_2 a_1, b_2 b_1. \] Without loss of generality,
\[ \text{(again up to reversal and rotation of } G), \text{ say that } j_1 \neq a_1. \]

We make the following subcases.

**Subcase 7.1:** We have \( s = 1. \)

There are two possibilities.

(i) \( i_1 \neq b_2. \) In this case \( G \) mimics MIN. 18. (See Figure 6.14.)

(ii) \( i_1 = b_2. \) If \( G \) has a vertex \( x \) between \( b_2 \) and \( a_1, \) then
\( G \) mimics MIN. 63. (See Figure 6.15.)

If \( G \) has no vertex between \( b_2 \) and \( a_1, \) then a reduction of
\( G \) is mimicked by MAX. 26. (See Figure 6.16.)

**Subcase 7.2:** We have \( s > 1. \)

By subcase 7.1, we may assume that \( i_2 = i_1 = b_2, \) and
thus that \( j_2 < j_1. \) In this case \( G \) mimics MIN. 67. (See Figure 6.17.)

This concludes our examination of subcase 7.

We next consider subcase 5. We may assume without
loss of generality that after iterated stripping $C_1$ and $C_2$ are two hump digraphs. Using rotation and reflection and subcase 7, we may say without loss of generality that one of the following two cases occurs:

**Subcase 5.1:** $G$ has extra-skeletal edges $a_2 a_1', b_2 b_1', a_2' a_1, b_2' b_1$, $i_1 j_1, i_2 j_2$ where $j_1 < a_1 < b_1 < a_2 < b_2 = i_1 < j_2 < a_1' < b_1' < a_2' < b_2' = i_2$.
In this case $G$ mimics MIN.63. (See Figure 6.18.)

**Subcase 5.2:** $G$ has extra-skeletal edges $a_2 a_1, b_2 b_1, a_2' a_1', b_2' b_1'$, $i_1 j_1, i_2 j_2$ where $j_1 = a_1 < b_1 < a_2 < b_2 < i_1 < j_2 < a_1' < b_1' < a_2' < b_2' = i_2$.
In this case $G$ mimics MIN.68. (See Figure 6.19.)

This concludes our examination of subcase 5, and hence our consideration of the case when $G$ has exactly two bubbles.

**Subcase:** $G$ has exactly one bubble:
Let the bubble be called $C_1$. If $C_1$ is a one hump digraph, then suppose that under $C_1$ we have two disjoint M's: viz. $G$ has edges $a_2 a_1, b_2 b_1, a_2' a_1', b_2' b_1', i_1 j_1, i_2 j_2$ where $j_2 < j_1 < a_1 < b_1 < a_2 < b_2 < a_1' < b_1' < a_2' < b_2' < i_1 < i_2$ and $d_1 = j_2, c_1 = i_2$. By rotation, assume that $d_1 = j_1$. 
In this case, $G$ mimics MIN.69. (See Figure 6.20.)

We may thus suppose without loss of generality that whenever $G$ is a digraph with only one bubble, by repeated iteration of the stripping procedure on $C_1$ we will eventually arrive at a digraph $C_1'$ which is a two or three hump digraph. This justifies the following case breakdown:

Subcase a: $C_1'$ is a three hump digraph.
Subcase b: $C_1'$ is a two hump digraph.

In pursuing subcase a, by our classification of the three hump case, we may assume without loss of generality that either (i) $C_1'$ falls under case A2(b), $j = b_1$ of the classification of the three hump digraphs or (ii) $C_1'$ is its own skeleton.

In subcase a (i), $G$ mimics MIN.70. (See Figure 6.21.)

In subcase a (ii), we make a distinction, depending on whether $\text{vert}(G) = \text{vert}(C_1')$.

If $\text{vert}(G) = \text{vert}(C_1')$ then a reduction of $G$ is mimicked by MAX.25. (See Figure 6.22.)
Figure 6.20
Figure 6.21

Figure 6.22

Figure 6.23
Otherwise, $G$ mimics MIN.71. (See Figure 6.23.)

We now turn to subcase $\beta$.

In this case, $G$ has (at least) certain edges $a_2a_1$, $b_2b_1$, $i_1j_1$, $i_2j_2$, ..., $i_sj_s$ where

$$d_1 = j_s \leq j_{s-1} \leq \ldots \leq j_1 \leq a_1 < b_1 < a_2 < b_2 < i_1 \leq \ldots \leq i_s = c_1.$$ Here $a_2a_1$, $b_2b_1$, along with the piece of $P$ from $a_1$ to $b_2$ form the skeleton of $C'_1$. The edges $i_jj_r$ are those that were stripped from $G$ to arrive at $C'_1$. We now make cases depending on the form of $C'_1$.

**Subcase- $C'_1$ is its own skeleton:** In this case, since edge $i_1j_1$ is not useless, we cannot have both $j_1 = a_1$ and $i_1 = b_2$. Without loss of generality (up to reflection), say that $i_1 \neq b_2$. We make further subdivisions based on the $ij$ edges.

**Subcase- $s = 1$:** In this case, a reduction of $G$ is mimicked by MAX.26. (See Figure 6.24.)

**Subcase- $s = 2$:** We have two possibilities. (i) $j_1 \neq a_1$. Without loss of generality, $i_1 \neq i_2$. In this case $G$ mimics MIN.18. (See Figure 6.25.)

(ii) $j_1 = a_1$. In this case (without loss of generality
up to a rotation, $j_2 = a_1$ also. Then a reduction of $G$ is mimicked by MAX.24. (See Figure 6.26.)

**Subcase—$s = 3$:** Because of our observations in the $s = 2$ case, we may now without loss of generality assume that $j_3 = j_2 = j_1 = a_1$. Then a reduction of $G$ is mimicked by MAX.24. (See Figure 6.27.)

**Subcase—$s > 3$:** Here $G$ mimics MIN.72. (See Figure 6.28.)

This finishes the case where $C'_1$ is its own skeleton. Returning to the theme of our stripping procedure, we use the two hump classification of the previous chapter on $C'_1$.

**Subcase—$C'_1$ has an edge of type D:** Here $G$ mimics MIN.35. (See Figure 6.29.)

From now on assume that $G$ has no edges of type D.

**Subcase—$C'_1$ has edges of type E only:** Here $C'_1$ must have two type E edges forming an M, and $C'_1$ mimics MIN.15 as we have already seen.
Figure 6.29

Figure 6.30
Subcase- $C'_1$ has edges of types A and E

only: We may assume by the foregoing subcase that $C'_1$ has no edge of type E. Thus $C'_1$ has two type A edges $k_1l_1$, $k_2l_2$ forming an M. By the case where $C'_1$ is its own skeleton and $j_1 \neq a_1$, assume that

$$a_1 = l_1 < l_2 < k_1 < k_2 < b_1.$$ (See Figure 5.22.) We make further subdivisions based on the $ij$ edges.

Subcase- $s > 1$: In this case, $G$ mimics MIN.66. (See Figure 6.30.)

Subcase- $s = 1$: If $i_1 \neq b_2$, then $G$ mimics MIN.66. (See Figure 6.31.)

We assume from here on that $i_1 = b_2$. If $C'_1$ has only two extra-skeletal edges, then a reduction of $G$ is mimicked by MAX.23. (See Figure 6.32.)

If $C'_1$ has a third extra-skeletal edge $k_3l_3$, we may assume without loss of generality, by the classification done in the previous chapter, that $l_3 = a_1$, $k_3 > k_2$. (See Figure 5.33.) In this case, $G$ mimics MIN.73. (See Figure 6.33.)
This completes those cases where all edges of $C_i'$ are of types A, E or D of the previous chapter.

**Subcase**—$C_i'$ has an edge of type B, but no type C edges: From our experience in the previous chapter, we may assume that every extra-skeletal edge of $C_i'$ is either a normal type B edge, or a reversed type B edge where the normal and reversed edges never cross to form any M. Let the extra-skeletal edges of $C_i'$ be

$$k_1a_1, k_2a_1, \ldots, k_pa_1, b_2l_1, b_2l_2, \ldots, b_2l_q$$

where $b_1 < k_1 < k_2 < \ldots < k_p < 1_q < 1_{q-1} < \ldots < 1_1 < a_2, p > q$.

We base our case division here on $p$, $q$ and $s$:

$q = 0$:

$s = 1$:

Assume (by rotation) that $a_1 = j_1$.

If $i_1 = b_2$, then the edge $i_1j_1$ is useless, a contradiction. (See Figure 6.34.)

In fact, for the case where every edge of $C_i'$ is a type B edge, we may assume that $a_1 = j_1, i_1 \neq b_2$.

If $i_1 \neq b_2$, then if $p \geq 3$, G mimics MIN.74. (See Figure 6.35.)

If $i_1 \neq b_2$ and if $p < 3$, then a reduction of G is mimicked by MAX.22. (See Figure 6.36.)
\( s > 1 \):
Again assume that \( j_1 = j_2 = a_1 \).

As \( i_1 \neq b_2 \), \( G \) mimics MIN.75. (See Figure 6.37.)

\( s > 0 \):
Assume that \( j_1 = a_1 \).
As \( i_1 \neq b_2 \), \( G \) mimics MIN.76. (See Figure 6.38.)
This finishes the case when \( C'_1 \) has no type C edges.

**Subcase- \( C'_1 \) has an edge of type C:** Call the type C edge \( k_l \). (Recall Figure 5.5.)

**Subcase- \( C'_1 \) has only one extra-skeletal edge:**

\( s > 2 \): Here \( G \) mimics one of MIN.18 or MIN.78, depending on whether \( k_l \) is a normal or reversed type C edge in \( C'_1 \). (See Figures 6.39 and 6.40.) Note that even if \( j_1 = a_1 \) and \( i_1 = b_2 \), \( i_1 j_1 \) contains the M formed by \( k_l \) and \( b_2 b_1 \), and is not useless.

\( s = 2 \): With only two \( i j \) edges, we may use reversal and rotation to assume that \( k_l \) is a normal type C edge, and \( j_1 = j_2 = a_1 \). If \( k < a_2 \), then \( G \) mimics MIN.79. (See Figure 6.41.)
Figure 6.39

Figure 6.40
If $i_1 > b_2$, then $G$ mimics MIN.18 (See Figure 6.42.)

If $k = a_2$ and $i_1 = b_2$ then a reduction of $G$ is mimicked by MAX.21. (See Figure 6.43.)

$s = 1$: Here a reduction of $G$ is mimicked by MAX.20. (See Figure 6.44.)

This concludes the case when $C_1'$ has only one extra-skeletal edge.

Subcase–$C_1'$ has two extra-skeletal edges:
Let the extra-skeletal edges of $C_1'$ be $k_11_1$, $k_22_2$. By the analysis of the previous chapter, we make the following case division.

$C_1'$ falls under $C_1(a)$: (Thus $C_1'$ is as depicted in Figure 5.59.) Here $G$ mimics MIN.80. (See Figure 6.45.)

$C_1'$ falls under $C_1(b)$: (Thus $C_1'$ is as depicted in Figure 5.65.) Here $G$ mimics MIN.81. (See Figure 6.46.)

$C_1'$ falls under $C_2(a)$: (Thus $C_1'$ is as depicted in Figure 5.63.) Here $G$ mimics MIN.82. (See Figure 6.47.)
Figure 6.41

Figure 6.42

Figure 6.43
C' falls under C2(b): (Thus C'₁ is as depicted in Figure 5.66.) We have two possibilities.

(i) \( i_s \neq b_2 \). Here G mimics MIN.83. (See Figure 6.48.)

(ii) \( i_s = b_2 \). By rotation we may assume that this means \( s = 1 \). Here a reduction of G is mimicked by MAX.19. (See Figure 6.49.)

C' falls under C3(a): (Thus C'₁ is as depicted in Figure 5.73.) Here G mimics MIN.84. (See Figure 6.50.)

C' falls under C3(b): (Thus C'₁ is as depicted in Figure 5.75.) We have two possibilities.

(i) \( i_s = b_2 \). By rotation we may assume that this means \( s = 1 \). Here a reduction of G is mimicked by MAX.18. (See Figure 6.51.)

(ii) \( i_s \neq b_2 \). Here G mimics MIN.85. (See Figure 6.52.)

C' falls under C4: (Thus C'₁ is as depicted in Figure 5.57.) Here G mimics MIN.86. (See Figure 6.53.)

C' falls under C5: (Thus C'₁ is as depicted in Figure 5.60.) We have two possibilities.
(i) $i_s \neq b_2$. Here $G$ mimics MIN.87. (See Figure 6.54.)

(ii) $i_s = b_2$. By rotation we may assume that this means $s = 1$. Here a reduction of $G$ is mimicked by MAX.17. (See Figure 6.55.)

**Subcase - $C'_1$ has three extra-skeletal edges:**

We may again fall back on the classification of the previous chapter. Also, by the foregoing section, assume that $i_1 = b_2$ and that $s = 1$.

**Subcase - $C'_1$ falls under case $C3(b) + C3(b)$:** (Thus $C'_1$ is as depicted in Figure 5.104.) Here $G$ mimics MIN.88. (See Figure 6.56.)

**Subcase - $C'_1$ falls under case $C3(b) + C2(b)$:** (Thus $C'_1$ is as depicted in Figure 5.107.) Here $G$ mimics MIN.89. (See Figure 6.57.)

**Subcase - $C'_1$ falls under case $C5 + C3(b)$:** (Thus $C'_1$ is as depicted in Figure 5.95.) Here $G$ mimics MIN.49. (See Figure 6.58.)

This concludes our proof.
Chapter 7: Lemmas On Substitutions

Let \( S = \{ x_1, x_2, x_3 \} \), \( T \) be alphabets. Let \( g: S^* \rightarrow T^* \) be a substitution. We produce certain conditions on \( g \) which are sufficient to show that \( g(h^n(x_2)) \) is non-repetitive where \( h \) is substitution 2.1. That is,

\[
\begin{align*}
    h(x_1) &= x_3, \\
    h(x_2) &= x_2x_3x_1, \\
    h(x_3) &= x_2x_1.
\end{align*}
\]

First we note that if the \( g(x_i) \) each start in a distinctive way, but have sufficiently different endings, then \( g \) works.

**Different Endings Lemma:** Let \( A, B \) be alphabets, \( T = A \cup B, A \cap B = \emptyset \). Suppose that for each \( i \), we can write

\[
g(x_i) = m b_i, m \in A, b_i \in B^* \]

so that the following conditions hold:

1. If for \( 1 \leq i, j, k \leq 3 \) we can write \( b_k = b_i b_j \) where \( b_i = b_i' b_i'' \), \( b_j = b_j' b_j'' \), then either \( b_i' = \epsilon, j = k \), or \( b_j' = \epsilon, i = k \). Thus we cannot glue together \( b_i \) from a prefix of \( b_j \) and a suffix of \( b_k \).
2. The word \( b_i \) is non-repetitive for each \( i \).

Then \( g(h^n(x_2)) \) is non-repetitive for all \( n \).

**Proof:** By Lemma 2.5, it suffices to show that the
following conditions hold:

1') If \( g(x) \) is a subword of \( g(y) \), where \( x, y \in S \), then \( x = y \).

3') If \( w \in S^* \) is a non-repetitive word, \( |w| = 3 \), then \( g(w) \) is non-repetitive.

Clearly condition 1') holds: If \( mb_i \) is a subword of \( mb_j \) we must have \( b_i \) a prefix of \( b_j \), whence \( i = j \), by condition 1).

It remains to show that condition 3') is fulfilled. Suppose \( g(x_1x_jx_k) \) contains a repetition for some \( i, j, k \), \( i \neq j \), \( j \neq k \). Thus \( mb_i mb_j mb_k \) contains a repetition \( vv \), \( v \neq \epsilon \). The word \( vv \) must contain exactly zero or two \( m \)'s. If \( vv \) contains no \( m \), then \( vv \) is a subword of \( b_j \) for some \( j \), contradicting 2). On the other hand, if \( vv \) contains the first two \( m \)'s of \( mb_i mb_j mb_k \), then \( b_i \) is a prefix of \( b_j \), impossible since \( i \neq j \).

Finally, if \( vv \) contains the last two \( m \)'s of \( mb_i mb_j mb_k \), then we can write \( v = b_i^m b_j^m b_k^m \), where \( b_i = b_i' b_i'' \), \( b_j = b_j' b_j'' \), \( b_k = b_k' b_k'' \). But then lining up the \( m \)'s, we get \( b_i'' = b_j'' \), \( b_j' = b_k' \), so that \( b_j = b_i' b_i'' \), contradicting 1). We conclude that \( g(x_1x_jx_k) \) is non-repetitive whenever \( i \neq j \), \( j \neq k \), so that \( g \) fulfills condition 3').
Block/Separator Lemma: Suppose that we can write
\[ g(x_i) = n_0 b_{i+1} n_b, \quad i = 1, 2, 3 \]
where \( b_i \in B^* \), each \( i \), some alphabet \( B \), \( n \in A \), some alphabet \( A \), such that \( A \cap B = \emptyset \). Suppose further that the following conditions are fulfilled:

1. If for \( 0 < i, j, k < 4 \) we can write \( b_k = b_i b_j \) where \( b_i = b'_i b''_i \), \( b_j = b'_j b''_j \), then either \( b'_i = \epsilon \), \( j = k \), or \( b''_j = \epsilon \), \( i = k \).

2. If \( 1 < i, j < 4 \), then \( |b_i| < |b_j| \).

3. The word \( b_i \) is non-repetitive, \( 0 \leq i \leq 4 \).

Then \( g(b^n(x_i)) \) is non-repetitive for all \( n \).

Proof: Our proof is analogous to the previous proof, but somewhat more involved. Again condition 1\textquoteleft \textquoteright of Lemma 2.5 will hold. It remains to show that condition 3\textquoteleft \textquoteright holds.

Suppose \( g(x_i x_j x_k) \) contains a repetition for some \( i, j, k, i \neq j, j \neq k \). Then we get
\[ nb_0 n_p n_b q n_0 n_r n_s n_0 n_t n_u \]
containing a repetition \( vv \), \( v \neq \epsilon \).

Case A: The word \( v \) contains \( n_0 n \) as a subword.
Examining \( g \), we see that \( vv \) must contain this subword exactly twice. If \( vv \) contains the first two occurrences of this subword in \( g(x_i x_j x_k) \) then \( b_p \) is a prefix of \( b_r \).
By condition (1), \( p = r \), so that \( i = j \), a contradiction.

Thus it must be the second two occurrences of \( n_0 n \)
which are in vv, and lining things up using these matching subwords in vv, we write
\[(bpnbq)'' = (brnb)'' \]
where as usual x' (x'') stands for a prefix (suffix) of x, and \[brnb = (brnb)''(brnb)''\]. But if n is in (brnb)''
then \[b_s = b_q\], so that \[s = q\], and \[i = j\], a contradiction. However, then n must be in (brnb)'
again giving a contradiction. We conclude that nb_0 n is not a subword of v.

**Case B:** The word vv contains a subword nb_0 n, but v does not.
Thus nb_0 n "straddles the border" between the two v's of vv, and we write \[v = Znb_0' = b_0''nY\], where \[b_0' b_0'' = b_0\]
and \[Z, Y \in (A \cup B)\]. Thus \[w = g(x_1 x_2 x_3)\] contains a subword of the form \[b_0''nXnb_0 nXb_0''\], \[X \in (A \cup B)\]. However, now by condition (1), we may assume that w contains either nb_0 nXnb_0 nX or Xnb_0 nXnb_0 n as a subword, and we are back in Case A, which has already been dealt with.

**Case C:** The repetition vv does not contain nb_0 n.
Thus without loss of generality, up to reindexing, assume that vv is a subword of \[b_0 nb_r nb_s nb_0\].
By condition (3), vv contains at least one n. Thus vv contains exactly two n's. If the first two n's here are in vv, then \[b_r\] is a prefix of \[b_s\], contradicting (1).
If the second two n's are in \( v v \), then we write \( b_s = b'_0 b''_r \) and get the usual contradiction.

We conclude that \( g(x_1, x_j, x_k) \) is non-repetitive. □

**Long/Short Lemma:** Suppose that we can write
\[
\begin{align*}
g(x_1) &= m b_1 e_1 \\
g(x_2) &= m b_2 e_1 b_1 e_2 \\
g(x_3) &= m b_2 e_2
\end{align*}
\]
where for each \( i, b_1, e_i \in A \), some alphabet \( A \), \( m \in B \), some alphabet \( B \), such that \( A \cap B = \emptyset \). Then if

1. Each of \( g(x_1), g(x_2), g(x_3) \) is non-repetitive
2. \( |b_1| < |b_2|, |e_1| < |e_2| \)
3. If \( w \) is a common prefix of \( b_2, b_1 \), \( y \) a common suffix of \( e_1, e_2 \), then \( |w v| < |b_1 e_1| \)
4. Any common prefix of \( b_1 e_i, b_2 e_j \) is of length \( \leq |b_1| \)
5. Any common suffix of \( b_1 e_i, b_2 e_2 \) is of length \( \leq |e_1| \)
then \( g \) is suitable.

**Proof:** We show that \( g \) is suitable, i.e.

1. \( |g(x_i)| \leq |g(x_j)| + |g(x_k)| \) for \( 1 \leq i, j, k \leq 3 \), \( i, j, k \) distinct
2. For \( 1 \leq i \leq 3 \), one cannot write \( g(x_i) = u w = w z \), \( u, w, z \) non-empty words over \( T \).
3. If \( w \in S^* \) is a non-repetitive word with \( |w| \leq 3 \), and \( w \neq x_1 x_3 x_1, x_2 x_3 x_2 \), then \( g(w) \) is non-repetitive.
Conditions 1) and 2) are easily checked. It remains to show that condition 3) holds. It will be useful to first consider the case when $|w| = 2$.

Let $w = x_1x_2$. Here $g(w) = mb_1 e_1 mb_2 e_1 mb_1 e_2$. Suppose that $vv$ is a subword of $g(w)$ for some $v \neq \varepsilon$. By condition (1), repetition $v$ must include the second $m$ of $g(w)$; any other repetition would be entirely inside $g(x_1)$ or $g(x_2)$. There are two possibilities:

Case 1: The first two $m$'s are in $vv$. In this case we must have $m$ for a prefix of $v$, so that $v = mb_1 e_1$. Therefore, $mb_1 e_1 = v$ is a prefix of $mb_2 e_1$. Thus $b_1 e_1$ is a prefix of $b_2 e_1$. This is a contradiction of condition (4), as $|b_1 e_1| > |b_1|$. Thus $g(w)$ is non-repetitive in this case.

Case 2: The second two $m$'s are in $vv$. Thus $vv$ is contained in the word $b_1 e_1 mb_2 e_1 mb_1 e_2$. Using the $m$'s to line up the pieces $v$, we have $(b_1 e_1)^w = (b_2 e_1)^w$, $(b_2 e_1)' = (b_1 e_2)'$ where $(b_1 e_j)'$ stands for a non-empty prefix of $b_1 e_j$, $(b_1 e_j)^w$ stands for a non-empty suffix of $b_1 e_j$, and $(b_2 e_1)'(b_2 e_1)^w = b_2 e_1$. However, by condition (4), $|b_2 e_1|' \leq |b_1|$. Lining up the $e_1$'s, we can therefore write $b_2 = b_1' b_1''$, where $b_1 = x b_1'' = \phi_1'y$ for some $x, y$. Since $|b_2| > |b_1|$, we can write $b_1'' = z y$, $b_1'$
= xz for some z ≠ ε. But then b₂ contains the repetition zz, a contradiction. (We call this an overlap argument.) Thus g(w) can have no repetition.

w = x₁x₃: In this case g(w) = mb₁e₁mb₂e₂. Any repetition vv involves both m's, and lining things up using the m's, we find that b₁e₁ is a prefix of b₂e₂, contradicting condition (4).

w = x₂x₁: Here g(w) = mb₂e₁mb₁e₂mb₁e₁. Any repetition vv must involve the second two m's, as the first two are contained in g(x₂). Then we get (b₂e₁)" = (b₁e₂)",
(b₁e₂)′ = (b₁e₁)′, and b₁e₂ = (b₁e₂)(b₁e₂)". Now by condition (5), (b₁e₂)" must actually be a suffix of e₁ alone. Lining up b₁'s, we get e₂ = e₁ e₁" and we use an overlap argument as in a previous case.

w = x₂x₃: We get g(w) = mb₂e₁mb₁e₂mb₁e₂. Any repetition involves the last two m's. We argue similarly to the previous case, except now we get | (b₁e₂)" | ≤ | b₁ |,
| (b₁e₂)" | ≤ | e₁ |. This forces | b₁e₂ | ≤ | b₁e₁ |, a contradiction.

w = x₃x₁: Here g(w) = mb₂e₂mb₁e₁. If there were a
repetition, we would have $mb_2e_2$ a prefix of $mb_1e_1$, which is absurd because of the respective lengths.

$w = x_3x_2$: We have $g(w) = mb_2e_2mb_2e_1mb_1e_2$. Any repetition must match the first two m's, forcing $b_2e_2$ to be a prefix of $mb_2e_1$, which is absurd.

We have thus established that $g$ behaves well on the two-letter words. It remains to consider the cases when $|w| = 3$:

$w = x_1x_2x_1$: Here $g(w) = mb_1e_1mb_2e_1mb_1e_2mb_1e_1$. Since the $g$ behaves well on two-letter words, any repetition $vv$ in $g(w)$ must straddle the images of all three letters here, thus containing at least the last three m's. We conclude that (since repetitions contain an even number of m's) all four m's are in $vv$. This implies that $mb_1e_1 = mb_1e_2$, which is absurd, as the lengths differ.

$w = x_1x_2x_3$: Here $g(w) = mb_1e_1mb_2e_1mb_1e_2mb_2e_2$, and we get the same contradiction as in the previous case. This contradiction will similarly occur for each $w$ of form $x_1x_2x_j$.

$w = x_1x_3x_2$: Here $g(w) = mb_1e_1mb_2e_2mb_2e_1mb_1e_2$. Here either all four
m's get matched up, or only the central two. If all four m's get matched, we get an absurdity involving lengths. We therefore assume that the second two m's are matched, and write \[(b_1 e_1)" = (b_2 e_2)" , (b_2 e_2)' = (b_2 e_1)'\] where \[(b_1 e_1)'\] stands for a non-empty prefix of \[b_1 e_1\], \[(b_1 e_1)"\] stands for a non-empty suffix of \[b_1 e_1\], and \[(b_2 e_2)'(b_2 e_2)"\] = \[b_2 e_2\]. However, by condition (5), \[| (b_2 e_2)" | \leq | e_1 |\]. Lining up the \(b_2\)'s we get a contradiction by an overlap argument.

\[w = x_2 x_1 x_2^-1\] Here \(g(w) = m b_2 e_1 m b_1 e_2 m b_1 e_1 m b_2 e_1 m b_1 e_2\) We match either the third and fourth m's, the first four m's, or the last four m's.

If the third and fourth m's are matched, after our usual argument we end up with \(b_1\) a prefix of \(b_2\), \(e_1\) a suffix of \(e_2\). This is forbidden by condition (3).

Matching the first four m's gives our standard absurdity involving length. Thus suppose the last four m's are matched. This forces \(b_1 e_2 = b_2 e_1\), contradicting the non-repetitiveness of \(g(x_2) = m b_2 e_1 m b_1 e_2\).

\[w = x_2 x_1 x_3^-1\] Here \(g(w) = m b_2 e_1 m b_1 e_2 m b_1 e_1 m b_2 e_2\). We cannot have all four m's in a repetition, as this gives the usual contradiction concerning lengths. The alternative is that the last two m's are matched by a repetition and here we get the same contradiction as in the previous
case.

$w = x_3 x_1 x_2$: Here $g(w) = mb_1 e_1 mb_1 e_2 mb_2 e_2 mb_1 e_1$. Involving all four $m$'s in a repetition is impossible, as usual. However, matching the last two $m$'s gives a contradiction by the overlap argument.

$w = x_3 x_1 x_3$: Here $g(w) = mb_1 e_2 mb_1 e_1 mb_2 e_2 mb_1 e_2$. Here either all four $m$'s get matched up, or only the middle two. If all four $m$'s get matched, we get an absurdity involving lengths. If the center two $m$'s are matched, then after our usual argument we end up with $b_1$ a prefix of $b_2$, $e_1$ a suffix of $e_2$. This is forbidden by condition (3).

$w = x_3 x_2 x_3$: Here $g(w) = mb_2 e_2 mb_1 e_1 mb_2 e_2$. The first two $m$'s cannot be paired. However matching the last two we end up with $b_1$ a prefix of $b_2$, $e_1$ a suffix of $e_2$. This is forbidden by condition (3).

$mb_1 e_1 = mb_1 e_2$, which is absurd, as the lengths differ.

$w = x_3 x_2 x_1$: As we remarked earlier the contradiction of the cases $w = x_1 x_2 x_1$, $x_1 x_2 x_3$ carries over to this case and the next.

$w = x_3 x_2 x_3$: See above.

Having looked at all the short words and finding $g$ to be well-behaved, we are finished our proof.
MIN.71: To deal with MIN.71, we use some substitutions on a five letter alphabet. Let $l: \{1, 2, 3, 4, 5\} \rightarrow \{1, 2, 3, 4\}$ be given by

1(1) = 43
1(2) = 4321
1(3) = 432153
1(4) = 41521
1(5) = 4153

Let $k: \{1, 2, 3, 4, 5\} \rightarrow \{1, 2, 3, 4, 5\}$ be given by

k(1) = 512343212345123212343243234
k(2) = 512343212345123212343243234512321234
k(3) = 5123432123451232123432345123212343243
k(4) = 51234321234512343234512321234324
k(5) = 51234321234512343234512321234324

We wish to show that $k(l^\omega(4))$ is non-repetitive. Consider the following simplified substitution lemma.

Lemma: Let $g: A^* \rightarrow B^*$ be a substitution so that

1. If $g(a_1 \ldots a_n) = X g(e) Y$, then for some $j$, $X = g(a_1 \ldots a_j)$, $a_{j+1} = e$. 


(2) If we can write \( g(x_i)^{''} = g(x_j)^{''} \), \( g(x_j)' = g(x_k)' \), \\
g(x_j) = g(x_j)'g(x_j)^{''} \) with \( x' ( x'' ) \) a prefix (suffix) 
of \( x \), then \( w = x_1x_jx_k \) is among \( w_1, w_2, \ldots, w_n \).

Then if \( v \) is a non-repetitive word never containing \( x_1x_jx_k \) where \( x_i^1 \ldots x_k^1 = w_r \), some \( r, 1 \leq r \leq n \), any word \( X \), then \( g(v) \) is non-repetitive. \( w_3, g(x_i^1x_j^1x_k^1) = |w| - 3 \), 
then \( g(w) \) is non-repetitive unless \( w = w_1, w_2, \ldots, w_m \).

This result follows from the proof of the substitution lemma. The "line-up" condition (1), will clearly be true of \( k \) and \( l \). The following list may be checked to suffice for condition (2) for substitutions \( k \) and \( l \): 123, 145, 154, 212, 213, 241, 242, 243, 245, 312, 313, 314, 315, 323, 345, 351, 352, 353, 354, 412, 413, 421, 423, 512, 513, 514, 515, 523, 532, 545.

Suppose that \( x_1^1x_j^1x_k^1 \) never appears in \( l^m(4) \) for any \( m \). Then \( l^m(4) \) is non-repetitive for each \( m \), by the simplified substitution lemma, and so is \( k(l^{w}(4)) \). Thus to show that \( k(l^{w}(4)) \) is non-repetitive it suffices to show that none of the listed triples occurs in this way in \( l^n(4) \). We now do this:

Suppose that for some \( m \), one of the listed triples occurs in the above manner in \( l^m(4) \). Choose \( m \) as small as
possible. In particular, $l^{m-1}(4)$ contains none of the above listed triples, so that $l^m(4)$ is non-repetitive.

**Triple 123:** Suppose that there exists a word $X = a_1a_2 \ldots a_r$ so that $1X2X3$ appears in $l^m(4)$. Then $a_1$ follows 2 in $l^m(4)$, so by examining 1, we conclude that $a_1$ is 1. However, then $l^m(4)$ contains 11, which, again examining 1, we see is impossible.

**Triple 254:** Suppose that there exists a word $X = a_1a_2 \ldots a_r$ and a letter $y$, so that $yX5X4$ appears in $l^m(4)$. Then $a_1$ follows 2, and must be a 1. Then $l^m(4)$ contains 51, which is impossible.

**Triples 212, 312, 412, 512:** Suppose that there exists a word $X = a_1a_2 \ldots a_r$ and a letter $y$, so that $yX1X2$ appears in $l^m(4)$. Then $a_r$ precedes 1, and must be a 2 or a 4. However $a_r$ precedes 2, and so must be a 3 or a 5. This is a contradiction.

**Triples 213, 313, 413, 513:** Suppose that there exists a word $X = a_1a_2 \ldots a_r$ and a letter $y$, $y = 2, 3, 4, 5$, so that $yX1X3$ appears in $l^m(4)$. Then $a_r$ is a 4, because it precedes both 3 and 1 in pieces from 1. Then $a_1$ follows 41, and must be a 5. However, $y$ precedes $a_1$ and thus must be a 1, contradicting our choice of $y$.

**Triple 314:** Suppose that there exists a word $X = a_1a_2 \ldots a_r$ so that $3X1X4$ appears in $l^m(4)$. Then $a_r$ precedes 4, and
is a 1 or 3. However neither 31 nor 11 can appear, so we have a contradiction.

**Triple 315**: Suppose that there exists a word $X = a_1 a_2 \ldots a_r$ so that $3X1X5$ appears in $l^m(4)$. Then $a_r$ precedes 5, and must be a 1. Then $l^m(4)$ contains 11, which is impossible.

**Triples 323**: Suppose that there exists a word $X = a_1 a_2 \ldots a_r$ so that $3X2X3$ appears in $l^m(4)$. Then $a_1$ follows 2 and must be a 1. But then $l^m(4)$ contains 31, which is impossible.

**Triples 351, 352, 353**: Suppose that there exists a word $X = a_1 a_2 \ldots a_r$ and a letter $y$, $y = 1, 2$ or 3, so that $3X5Xy$ appears in $l^m(4)$. Then $a_r$ precedes 5, and must be a 1. This forces $y$ to be a 4 or a 5, contradicting our choice of $y$.

**Triple 421**: Suppose that there exists a word $X = a_1 a_2 \ldots a_r$ so that $4X2X1$ appears in $l^m(4)$. Then $a_r$ precedes 2, and is a 3 or a 5. However, as $a_r$ precedes 1, it must be a 4 or a 2. This is a contradiction.

**Triple 423**: Suppose that there exists a word $X = a_1 a_2 \ldots a_r$ so that $4X2X3$ appears in $l^m(4)$. Then $a_1$ precedes 2, and must be a 1. Now $a_2$ is preceded by 41, and must be a 5. Thus $a_r$ is followed by 215, and is a 3. But then $l^m(4)$ contains 33, which is a contradiction.

**Triples 513, 514, 515**: Suppose that there exists a word
$X = a_1a_2..a_r$ and a letter $y$, so that $5X1Xy$ appears in $l^m(4)$. Then $a_1$ follows 1 and must be 5 or 4. However, $a_1$ follows 5, and must be 3 or 2. This is a contradiction.

**Triples 523:** Suppose that there exists a word $X = a_1a_2..a_r$ so that $5X2X3$ appears in $l^m(4)$. Then $a_1$ follows 2 and must be 1, leaving $l^m(4)$ to contain 51, which is impossible.

**Triple 532:** Suppose that there exists a word $X = a_1a_2..a_r$ so that $5X3X2$ appears in $l^m(4)$. Then $a_1$ follows 5 and 3, and is forced to be a 2. Thus $a_r$ must be a 4, as it is followed by 32. But then $l^m(4)$ contains 42, which is impossible.

**Triples 145, 245, 345, 545:** Suppose that there exists a word $X = a_1a_2..a_r$ and a letter $y$, so that $yX4X5$ appears in $l^m(4)$. Then $a_r$ precedes 5, and must be a 1. If $r = 1$, then $l^m(4)$ contains 141, so that $l^{m-1}(4)$ contains one of 24, 25, 44, 45, which is impossible, since 1 never produces these words. Thus $r > 1$. Since $a_1$ follows 4, $a_1$ is 3 or 1. However $a_1$ cannot be 1, or $l^m(4)$ contains 141.

Now $y$ precedes 3, and must be 4 or 5. However, if $y = 4$, then $l^m(4)$ contains the repetition 4X4X, contradicting the minimality of $m$. Thus $y = 5$. Since $a_2$ follows 53, $a_2 = 4$.

Continuing these kinds of arguments, it may be shown
that in fact $l^m(4)$ contains a block of the form

$$53 41521 Z 1^2(4) 41521 Z 1^2(5).$$

However the block $41521 Z$ is of the form $1^2(Q)$. We thus deduce that $l^{m-2}(4)$ contains $Q4Q$. This contradicts the minimality of $m$.

**Triples 241, 242, 243:** Suppose that there exists a word $X = a_1a_2...a_r$ and a letter $y$, $y \neq 4, 5$, so that $2X4Xy$ appears in $l^m(4)$. Then $a_1$ must be a 1 as it follows 2 and 4. Then $a_2$ follows 41, and is a 5, $a_3$ follows 215, and is a 3. Thus $a_3$ follows 2153, and is a 4.

As $a_r$ is followed by a 4, $a_r$ is 3 or 1. But $a_r$ is not 1, otherwise $l^{m-1}(4)$ would contain one of 24, 25, 44, 45, which is impossible. Thus $a_r$ is a 3. Then $y$ is a 4 or 2. However, $y$ cannot be a 4, or $l^m(4)$ contains the repetition $X4X4$, which contradicts the minimality of $m$.

Continuing these kinds of arguments, it may be shown that in fact $l^m(4)$ contains a block of the form

$$1^2(2) 41521 Z 1^2(4) 41521 Z 41521432.$$  

However the block $41521 Z$ is of the form $1^2(Q)$. We thus deduce that $l^{m-2}(4)$ contains $2Q4Q$. This contradicts the minimality of $m$.

Since none of the bad triples occur, $k(l^m(4))$ is non-repetitive for every $m$.
Chapter 8: Substitutions

We wish to show that each of the graphs of MIN is versatile. We saw at the end of Chapter 2 that MIN.1 - MIN.4 are versatile. MIN.71 is treated separately in Chapter 7. For each other digraph G of MIN, we give a substitution f meeting the demands of the Substitution Lemma (Lemma 2.4), such that f(g(2)) is a non-repetitive walk of type ω on G, where g is substitution 2.1.

The conditions of the Substitution Lemma are such that they are easily verified for each of these substitutions by computer. However, to aid the understanding of the reader, we label substitutions with the labels 'D/E', 'B/S', 'L/S' standing for 'Different/Endings', 'Block/Separator' and 'Long/Short' respectively. A substitution with such a label falls into the corresponding category of substitutions as discussed at the end of Chapter 3, and can usually be shown to be suitable using the corresponding theorem of Chapter 7.

MIN.5: x₁: 12342
      x₂: 1234542123423 L/S
      x₃: 12345423

212
| MIN. 6: | $x_1$: 1234543 |
|MIN. 7: | $x_1$: 12345423 |
|MIN. 8: | $x_1$: 1234523 |
|MIN. 9: | $x_1$: 1343 |
|MIN. 10: | $x_1$: 13423 |
|MIN. 11: | $x_1$: 1434543 |
|MIN. 12: | $x_1$: 14542 |
|MIN. 13: | $x_1$: 12343 |
| MIN. 14: | $x_1$: 123454 |
|         | $x_2$: 123212345412345654 | L/S |
|         | $x_3$: 123212345654 |
| MIN. 15: | $x_1$: 12345234 |
|         | $x_2$: 123432345234123452343234 | L/S |
|         | $x_3$: 1234323452343234 |
| MIN. 16: | $x_1$: 123454 |
|         | $x_2$: 12343234541234543234 | L/S |
|         | $x_3$: 12343234543234 |
| MIN. 17: | $x_1$: 1234565452 |
|         | $x_2$: 123456523452123456545232 | L/S |
|         | $x_3$: 12345652345232 |
| MIN. 18: | $x_1$: 12345 |
|         | $x_2$: 123432345123456 | L/S |
|         | $x_3$: 1234323456 |
| MIN. 19: | $x_1$: 12345674 |
|         | $x_2$: 123212345674123456765674 | L/S |
|         | $x_3$: 1232123456765674 |
| MIN. 20: | $x_1$: 12321234567 |
|         | $x_2$: 123456545671232123456787 | L/S |
|         | $x_3$: 1234565456787 |
| MIN. 21: | $x_1$: 123212345 |
|         | $x_2$: 12343234512321234565 | L/S |
|         | $x_3$: 12343234565 |
MIN.22:  \( x_1: 123534 \)
\( x_2: 1234534123534234 \) L/S
\( x_3: 1234534234 \)

MIN.23:  \( x_1: 1234212345 \)
\( x_2: 12342312345123421234565 \) L/S
\( x_3: 1234231234565 \)

MIN.24:  \( x_1: 123432345 \)
\( x_2: 123431234512343234565 \) L/S
\( x_3: 123431234565 \)

MIN.25:  \( x_1: 123212345676123456 \)
\( x_2: 123212345612345 \) B/S
\( x_3: 1232123451234 \)

MIN.26:  \( x_1: 1232123451234 \)
\( x_2: 1232123456412345 \)
\( x_3: 123212345645 \)

MIN.27:  \( x_1: 1234565 \)
\( x_2: 1232123456512345645 \) L/S
\( x_3: 123212345645 \)

MIN.28:  \( x_1: 123453 \)
\( x_2: 12345231234534 \) L/S
\( x_3: 12345234 \)

MIN.29:  \( x_1: 12345621234565 \) L/S
\( x_2: 123456231234565123456212345645 \)
\( x_3: 1234562312345645 \)
MIN. 30: \[ x_1: \text{adac}b \]
\[ x_2: \text{ad}c \]
\[ x_3: \text{ae}b \]
where \( a = 1234565 \)
where \( b = 1234562 \)
where \( c = 12345623 \)
where \( d = 123456234 \)
where \( e = 1234562345 \)

MIN. 31: \[ x_1: 12342 \]
\[ x_2: 12345342123423 \]
\[ x_3: 123453423 \]

MIN. 32: \[ x_1: 12342312345342345323123453423123453 \]
\[ x_2: 12342312345342345312345323423123453423453 \]
\[ x_3: 1234231234534234532342312345323423 \]

MIN. 33: \[ x_1: 12342312345342341234534 \]
\[ x_2: 123423123453423453123453234534234 \]
\[ x_3: 12342312345341234534 \]

MIN. 34: \[ x_1: 1234534 \]
\[ x_2: 12321234534123453234 \]
\[ x_3: 1232123453234 \]

MIN. 35: \[ x_1: 123423123453423453 \]
\[ x_2: 12342312345342345123453123453123453 \]
\[ x_3: 123453423453123453 \]
\[
x_3 : 1234231234534234512345312345
\]

MIN. 36: \( x_1 : 12345634 \)
\[
x_2 : 1234563234123456345 \quad \text{L/S}
\]
\[
x_3 : 12345632345
\]

MIN. 37: \( x_1 : 12345643452345645 \)
\[
x_2 : 123456452345643452345645123456434523456452345
\]
\[
x_3 : 1234564523456434523456452345
\]

MIN. 38: \( x_1 : 123454234 \)
\[
x_2 : 1234534234123454234534 \quad \text{L/S}
\]
\[
x_3 : 123453423454
\]

MIN. 39: \( x_1 : 12345434 \)
\[
x_2 : 123454234123454342 \quad \text{L/S}
\]
\[
x_3 : 1234542342
\]

MIN. 40: \( x_1 : 1234567345 \)
\[
x_2 : 123456732345123456734565 \quad \text{L/S}
\]
\[
x_3 : 12345673234565
\]

MIN. 41: \( x_1 : 2123454345 \)
\[
x_2 : 234123454345212345412345 \quad \text{L/S}
\]
\[
x_3 : 2341234543412345
\]

MIN. 42: \( x_1 : 1232123456765612345676 \)
\[
x_2 : 123212345676123456 \quad \text{B/S}
\]
\[
x_3 : 12321234561234
\]

MIN. 43: \( x_1 : 1232123454 \)
\[
x_2 : 123421234541232123454234
\]
| x̄₃: 12342123454234 |
|---|---|
| **MIN.44**: x₁: 1234564541234563234 | L/S |
| x₂: 123456345412345632341234564541234563234564 |
| x₃: 12345634541234563234564 |
| **MIN.45**: x₁: 12345631234565 |
| x₂: 1234563123456345 |
| x₃: 12345632345 |
| **MIN.46**: x₁: 1234564345631234564345623456434565 |
| x₂: 123456434562345643456234565123456434565 |
| x₃: 12345632345 |
| **MIN.47**: x₁: 12342123423423 |
| x₂: 123432123423 | L/S |
| x₃: 1234323 |
| **MIN.48**: x₁: 123456234512345643456234564345623456434562 |
| 1234563456234565 | B/S |
| x₂: 12345623451234564345643456234565123456434565 |
| x₃: 1234562345123456434565123456434565 |
| **MIN.49**: x₁: 12345634 |
| x₂: 1234562341234563454 | L/S |
| x₃: 12345623454 |
| **MIN.50**: x₁: 1234564345 |
| x₂: 12345642345123456434565 | L/S |
| x₃: 1234564234565 |
| **MIN.51**: x₁: 12345632345 |
\[ x_2 : 1234563423451234563234565 \quad L/S \]
\[ x_3 : 12345634234565 \]

**MIN. 52:**
\[ x_1 : 12345234123454345234543412345434534 \]
\[ x_2 : 12345234123454345234541234534 \]
\[ x_3 : 1234523412345434523453412345434 \]

**MIN. 53:**
\[ x_1 : 1234543423454 \]
\[ x_2 : 123454345234541234543423454234 \]
\[ x_3 : 1234543452345434 \]

**MIN. 54:**
\[ x_1 : 123456534523456545 \]
\[ x_2 : 12345654534565234 L/E \]
\[ x_3 : 123456545234565345 \]

**MIN. 55:**
\[ x_1 : 1234534 \]
\[ x_2 : 12343234534123453234 L/S \]
\[ x_3 : 1234323453234 \]

**MIN. 56:**
\[ x_1 : 123454 \]
\[ x_2 : 12321234541234564 \quad L/S \]
\[ x_3 : 12321234564 \]

**MIN. 57:**
\[ x_1 : 1234567323456761234567456 \]
\[ x_2 : 12345673234567561234567456 \]
\[ 12345673234567612345673456 \quad L/S \]
\[ x_3 : 123456732345675612345673456 \]

**MIN. 58:**
\[ x_1 : 1234564512345632345623451234563234565 \]
\[ x_2 : 12345634512345632345623451234563234565 \]
\[ 12345645123456323456234512345632345645 \]
12345623451234563234565 L/S

x_3: 123456345123456323456234512345632345645-
12345623451234563234565

MIN.59: x_1: 1234563423456832345
x_2: 1234563452345634234562345 D/E
x_3: 123456345234563234562345

MIN.60: x_1: 2345612345673456234567345
x_2: 2345612345673452345673456734 B/S
x_3: 234561234567342345673

MIN.61: x_1:1232123456789
x_2:123456545678912321234567898789 L/S
x_3:12345654567898789

MIN.62: x_1: 12321234567
x_2: 1234321234567123212345676567 L/S
x_3: 12343212345676567

MIN.63: x_1: 23456781
x_2: 234323456781234567876781 L/S
x_3: 2343234567876781

MIN.64: x_1: 123421234567
x_2: 12342312345671234212345676567
x_3: 12342312345676567

MIN.65: x_1: 123431234567
x_2: 123432312345671234312345676567
x_3: 123432312345676567
MIN.66: $x_1: 1232$
$x_2: 123454$ D/E
$x_3: 123456$

MIN.67: $x_1: 123456787678567812345678767845678$
$L/S$
$x_2: 1232123456787678567812345678767845678$
$x_3: 123212345678767845678$

MIN.68: $x_1: 12345678$
$x_2: 1232123456787678$
$x_3: 123412321234567876785678$

MIN.69: $x_1: 123456$
$x_2: 123212345654561234567$
$x_3: 123212345654567$

MIN.70: $x_1: 123212345$
$x_2: 123421234512321234542345$ L/S
$x_3: 123421234542345$

MIN.72: $x_1: 1232123451234$
$x_2: 123212345612345$ B/S
$x_3: 12321234567123456$

MIN.73: $x_1: 12321234565123456$
$x_2: 123212345612345$
$x_3: 1232123451234$

MIN.74: $x_1: 123456712345623451234562345$
$x_2: 123456712345623412345623$ B/S
$x_3: 1234567123456231234562$
MIN. 75: $x_1: 1234212345$
$x_2: 1234231234512342123456$ L/S
$x_3: 123423123456$

MIN. 76: $x_1: 12342$
$x_2: 12343$ D/E
$x_3: 12345$

MIN. 77: $x_1: 123453$
$x_2: 12343234531234532343$ L/S
$x_3: 12343234532343$

MIN. 78: $x_1: 1234212345$
$x_2: 12342321234512342123456$ L/S
$x_3: 1234232123456$

MIN. 79: $x_1: 1234534$
$x_2: 123456$ D/E
$x_3: 1234532345$

MIN. 80: $x_1: 12345434$
$x_2: 123454234123454345$ L/S
$x_3: 1234542345$

MIN. 81: $x_1: 123453234$
$x_2: 12345342341234532345$ L/S
$x_3: 12345342345$

MIN. 82: $x_1: aec$
$x_2: afcaecb$ L/S
$x_3: afcb$
where

\[ a = 12345234 \]
\[ b = 12345434 \]
\[ c = 123454345 \]
\[ e = 12345434523454 \]
\[ f = 1234543452345434 \]

**MIN.83:**

- \( x_1: 12345 \)
- \( x_2: 123432345123423 \) L/S
- \( x_3: 1234323423 \)

**MIN.84:**

- \( x_1: 123456123454 \)
- \( x_2: 12345632341234563454 \)
- \( x_3: 123456323456123456323454 \)

**MIN.85:**

- \( x_1: 123454 \)
- \( x_2: 12345323454123456 \) L/S
- \( x_3: 12345323456 \)

**MIN.86:**

- \( x_1: 1232123432341234323 \)
- \( x_2: 1232123432312343 \) B/S
- \( x_3: 1232123431234 \)

**MIN.87:**

- \( x_1: 123456 \)
- \( x_2: 1234534 \) D/E
- \( x_3: 12345323 \)

**MIN.88:**

- \( x_1: 12345632345612345645 \) L/S
- \( x_2: 123456323456512345645123456323456123456345 \)
- \( x_3: 1234563234565123456345 \)
MIN.89: $x_1$: afd
$x_2$: bfdafecd  L/S
$x_3$: bfecd

where $a = 123454$
$b = 1234534$
$c = 12345234$
$d = 1234532345$
$e = 12345323454$
$f = 1234532345234$
Chapter 9: Non-versatility of MAX

It is the purpose of the present chapter to show that none of the digraphs of MAX are versatile. We commence by proving a useful theorem. First we make a definition:

**Definition:** Suppose that \( v \) is a non-repetitive word of type \( \omega \), on alphabet \( \Sigma = \{ a_1, a_2, \ldots, a_n \} \). An \( a_1 \)-block is a subword \( w \) of \( v \) so that \( a_1 \) is a prefix of \( w \), \( w \) contains exactly one \( a_1 \), and \( w \) appears in the context \( wa_1 \) in \( v \).

**Block Theorem:** Let \( a, b, c, d \) be words over some alphabet \( \Sigma \) such that \( b \) is a prefix of \( c \), which is a prefix of \( d \). Then \( a, b, c, d \) cannot be concatenated to form a non-repetitive word of type \( \omega \).

**Proof:** Suppose that \( a, b, c, d \) could be concatenated to form a non-repetitive word \( v \) of type \( \omega \). Suppose that the word \( a \) does not appear in \( v \) infinitely often; then word \( b \) never occurs, as \( bc \) and \( cd \) contain repetitions. But then \( c \) never occurs, for \( c \) cannot be followed by \( d \) in a non-repetitive word. This leaves the single word \( d \), which of course cannot be concatenated with itself to form any non-repetitive words.

Thus we may assume that \( v \) contains the word \( a \). Since \( v \) is an \( \omega \) word, assume without loss of generality that \( v \)
commences with the word $a$. In fact assume without loss of
generality that every one of the words $a$, $b$, $c$, $d$
appearing in \( v \) occurs infinitely often in \( v \). We may think
of \( v \) as a "meta-word", whose letters are $a$, $b$, $c$, $d$. If
we parse \( v \), chopping it into pieces at each occurrence of
$\,a$, the possible $a$-blocks are:

- $ab$
- $ac$
- $B: ad$
- $A: acb$
- $D: adc$
- $C: adb$
- $E: adcb$

We never, of course, find subwords $bc$, $bd$, $cd$ in $v$, as
these contain repetitions. Moreover, of these $a$-blocks,
only $A$, $B$, $C$, $D$, and $E$ can appear infinitely often in $v$;
the piece $ab$ is a prefix of the other pieces, and thus
never appears in $v$. (What would follow it in $v$? ) Again,
once we have disposed of piece $ab$, piece $ac$ is a prefix
of all the other pieces and cannot be used either. Thus $v$
is concatenated from pieces $A$, $B$, $C$, $D$, $E$. We assume
without loss of generality that each of these pieces
appearing in $v$ does so infinitely often.

The eccentric lettering of these pieces ($B$, $A$, $D$,
C, E) simply makes note of the fact that B is a prefix of C, which is a prefix of D, a prefix of E. We now take our argument one level deeper; as B is a prefix of C, which is a prefix of D, a prefix of E, v must contain block A. Parse v by chopping it up wherever the piece A appears followed by an a. Offhand, we get several pieces. However some of these A-blocks can only appear finitely often in v, and can hence without loss of generality, be assumed not to occur in v.

AB (1), AC (1), AD (4), AE

AED (2), AEC (2), AEB (2), ADC, ADB, ACB (3)

AEDC (2), AEDB (2), AECB (2), ADCB

AEDCB (2)

Notes: (1) As the block AB is a prefix of all the other blocks, it cannot appear in v. However, the block AC is a prefix of every block but AB, and hence AC cannot appear in v either.

(2) Here AEX (where X is B, C or D) will contain cb adcb ad, a repetition. Thus no block containing such a
word can appear in $v$.

(3) Here $ACB$ contains the repetition badbad.

(4) After the eliminations of (1) and (3), $AD$ is a prefix of the remaining blocks, and must be discarded.

We are left with four $A$-blocks to concatenate to form $v$:

$\alpha$: AE

$\beta$: ADB

$\gamma$: ADC

$\delta$: ADCB

We have almost come full circle; here $\beta$ is a prefix of $\gamma$, a prefix of $\delta$. Again $\alpha$ must appear in $v$. However, we now have quite a lot of conditions on $\alpha$, $\beta$, $\gamma$, $\delta$. By our examination of $A$-blocks, we know that the blocks resulting when $v$ is chopped into pieces at $\alpha$ are:

\[\begin{align*}
\alpha \delta \\
\alpha \gamma \\
\alpha \delta \gamma \\
\alpha \delta \beta \\
\alpha \delta \gamma \beta
\end{align*}\]

However here $\alpha \gamma \beta > E$ ADC ADB > bADadbADad, a repetition, so that the block $\alpha \gamma \beta$ can never appear in $v$. However,
once $a_1 \beta$ has been discarded, $a_1 \delta$ is a prefix of the four remaining blocks and must also be discarded. This leaves three blocks, $a_1 \delta \beta$, $a_1 \delta \gamma$, $a_1 \delta \gamma \beta$, with the first a prefix of the other two. One checks that there are no non-repetitive words of length greater than three on two letters. Since $v$ therefore could not be formed from two blocks, we have a contradiction. Thus words $a$, $b$, $c$, $d$ cannot be concatenated to form a non-repetitive word of type $w$. □

Using this Block Theorem, and similar arguments, we show that none of the digraphs of MAX is versatile.

MAX. 1

Suppose that we could walk some non-repetitive word $v$ of type $\omega$ on MAX. 1. If $v$ contains no 2, then $v$ can be walked on one of the strongly connected components of MAX. 1 \ { 2 }. However none of these components has more than two vertices, so that this is impossible.

Parse $v$ by chopping it into pieces commencing with 2. The possible 2-blocks on MAX. 1 are:

- a: 21
- b: 23
- c: 2345
By the Block Theorem, these words cannot be concatenated to form \( v \). This is a contradiction and we conclude that \( \text{MAX}.1 \) is not versatile.

\textbf{MAX}.2

The proof that \( \text{MAX}.2 \) is not versatile is more involved. Suppose that \( \text{MAX}.2 \) is versatile, and let \( v \) be a non-repetitive word of type \( \omega \) walkable on \( \text{MAX}.2 \). If \( v \) contains no 3, then \( v \) is walked on one of the strongly connected components of \( \text{MAX}.2 \setminus \{ 3 \} \), which is impossible, as each of these components consists of a single vertex. By analyzing the 3-blocks of \( v \) which could be walked on \( \text{MAX}.2 \), we can conclude that \( \text{MAX}.2 \) is not versatile.

\textbf{Level 1: 3-blocks:}

\begin{itemize}
  \item a: 34567
  \item b: 345612
  \item c: 3452
  \item d: 342
  \item e: 32
\end{itemize}

One checks that these are all the 3-blocks on \( \text{MAX}.2 \).
Next, looking at \( v \) as composed of letters \( a, b, c, d, e \), we look at \( b \)-blocks. (Note that \( v \) must contain a \( b \); otherwise \( v \) is composed of blocks \( a, c, d, e \). This possibility is not excluded by the Block Theorem above, however the proof that it cannot occur follows almost exactly the proof of the Block Theorem, and is therefore here omitted.) Here are the \( b \)-blocks of which \( v \) could be composed:

**Level 2: \( b \)-blocks:**

\[
\text{ba, } \text{bc (1), } \text{bd (1), } \text{be (1)}
\]

The underlined two-letter words never appear in \( v \), as will be shown in note (1) of the comments below. We therefore omit looking at any blocks on three or more letters that contain these words. Call a word which cannot appear in \( v \) more than finitely often **illegal**. Clearly no block containing an illegal word can appear in \( v \) more than finitely often, so that such blocks may be discarded without loss.

\[
\text{bac, bad, bae}
\]

\[
\text{baca, } \text{abcd (1), } \text{bace (1), bada, badc (1),}
\]
bade (1), bae (2), baec (1), baed (1)

Again, the underlined blocks are illegal, and this decreases the number of blocks on five or more letters we need to examine.

bacac (3), bacad, baca, badac (4), badad (5), badae

bacada, bacadc (1), baced (1), baca (2), bacaec (1), baca (1), badae (2), bacec (1), baced (1), badae (1), badaed (1)

bacadac (4), bacad (5), baca

bacadaea (2), bacadaec (1), baced (1)

Notes on the b-blocks

(1) The words 2e3, 2d34, 2c345 are repetitions, hence illegal. Since each b-block starts with 3, we therefore see that b-blocks containing the piece 2e may be discarded. Thus the words de, ce, be are illegal. This means in particular that d, whenever it appears in v, is
always followed by a b-block commencing 34.

It follows that ed, cd, bd are illegal.

As cd and ce are illegal, c is always followed by 345 in any b-block. Thus ec, dc, bc are illegal.

(2) The particle aea is always preceded by a 2 and followed by a three, and thus contains 2a3 2a3. Thus aea is illegal.

(3) Here acac is a repetition. We thus discard any block containing acac.

(4) The block contains 2a34 2a34.

(5) Here adad is a repetition.

Summary of usable b-blocks

Let us here list the b-blocks which we have not yet eliminated:

ba (1), bac, bad (6), bae (5), baca (3), bada (2), bacad, baca, badae, bacada (4), bacadae

Again, many of these blocks must be discarded.

(1) The block ba is a prefix of all the other useful b-blocks, and thus can’t be followed by any of them in a
non-repetitive word.
(2) Leads to bada $34 \Rightarrow 2\text{a}34 2\text{a}34$.
(3) Gives baca $345 \Leftarrow 2\text{a}345 2\text{a}345$
(4) Here this block appears only in the context bacada $34$ which contains $2\text{a}34\text{a}34 = 2\text{a}34 2\text{a}34$.
(5) After the elimination of $\text{ba}$, baca, bada, bacada, the only context in which this block could appear is $2\text{bae}$ $\text{ba}3 = 2\text{ba}3 2\text{ba}3$.
(6) Now this block appears only in context $2\text{badba}34 = 2\text{ba}34 2\text{ba}34$.

We are left with five b-blocks:

A: bac
B: badae
C: bacad
D: baca
+: bacadae

After one more level of blocks, we are done. Note that $A < C < D < E$ in the sense that the words AC, AD, AE, CD, CE, DE are illegal, so that v must contain a B.

Level 3: B-blocks:
BA, BC, BD, BE,
BAC (1), BAD (1), BAE (1), BCA, BCD, BCE (2), BDA (3),
BDC (3),
BDE (3), BEA (4), BEC (4), BED (4)

BCAC (1), BCAD (1), BCAE (1), BCDA, BCDC (5), BCDE (5)

BCDAC (1), BCDAD (1), BCDAE (1)

Notes on the B-blocks

(1) A is a prefix of C, D, E.
(2) C is a prefix of E.
(3) Here BDbac > adaebac adaebac, so that BDX is illegal,
where X is A, C or E.
(4) BEbac > adaebac adaebac.
(5) Block contains CDbacad > 2baca3 2baca3.

Summary of useful B-blocks

BA (1)
BC (4)
BD (2)
BE (3)
\( \alpha: \) BCA

\( \beta: \) BCD

\( \gamma: \) BCDA

Notes: (1) Must be discarded, since it is a prefix of the others.

(2) Once (1) is gone, this block always appears in the context 2BDBbaca \( \Rightarrow \) 2Bbaca3 2Bbaca3.

(3) This word EBb is illegal, as it contains adaeb adaeb.

(4) After (1), (2), (3) are gone, this block is a prefix of the remaining blocks.

We are thus left with blocks \( \alpha, \beta, \gamma \) with which to form a non-repetitive word of type \( \omega \). However, as \( \beta \) is a prefix of \( \gamma \), \( \beta \gamma \) contains a repetition. Recall from our remarks in Chapter 1 that if \( \alpha, \beta, \gamma \) can be concatenated to form a non-repetitive \( \omega \) word, then \( \alpha \beta, \alpha \gamma, \beta \alpha, \beta \gamma, \gamma \alpha, \gamma \beta \) must each be non-repetitive, We thus conclude that MAX.2 is not versatile.

Having given some details for MAX.1, MAX.2, we give less detail for the other cases, as there are, after all, 26 digraphs in MAX.
Suppose that $v$ is a non-repetitive word on MAX.3. Then $v$ must contain a 3, since MAX.3 \ {3} has only trivial strongly connected components.

**Level 1: 3-blocks:**

- a: 32
- b: 342
- c: 34512
- d: 3456
- e: 34562

Suppose that $v$ contains no d. Then a occurs only as 2a3, a repetition. Thus $v$ contains no a. With a, d excluded, b must occur as 2b34, a repetition, so that $v$ contains no d, b or a. This is impossible.

**Level 2: d-blocks:**

- da, db, dc, de (2)

- dab (1), dac, dae, dba (1), dbc, dbe, dca (1), dcb (1), dce

- daca (1), dacb (1), dace, daea (1), daeb (1), daec, dbca (1), dbcb (1), dbce, dbea (1), dbeb (1), dbec,
Notes on the d-blocks

(1) The word 2a3 is a repetition. Since each d-block starts with 3, we therefore see that the combinations ba, ca, da cannot appear. This means that b, whenever it appears in a non-repetitive of type w, is always followed by a block commencing 34. However, 2 b 34 is a repetition. Thus blocks ab, cb, eb must not be used.

(2) The block d is a prefix of e.

(3) Here ce repeats.

(4) The block contains ec ec.

Summary of useful d-blocks
(1) Leads to 2 da d3 = 2d32d3, a repetition.
(2) As the word 2e d is a repetition.
(3) Since 2ec d > 2cd 2cd, a repeat.
(4) After eliminations (1) - (3), all remaining blocks except for db end in c. Thus dc occurs in context 2 db dc > 2d342d34, or in context c dc d, both repetitions.
(5) As in elimination (4), this block is preceded either by db or c, giving rise to word 2 db dc, which contains a repetition, or c dce > cdcd.

We are left with five d-blocks:

A: db
B: dbc
C: dbec
D: dac
E: daec

After one more level of blocks, we are done. If D does not occur in v, then v is concatenated from A, B, C, E, and A < B < C in the sense that AB, AC, BC are illegal. Here BC is illegal because it must occur in the context cBC > cdbcdb. Arguing analogously to the proof of the Block Theorem, we get a contradiction.

Level 3: D-blocks:

DA, DB, DC, DE

DAB (1), DAC (1), DAE, DBA (1), DBC (1), DBE, DCA, DCB, DCE (1), DEA, DEB, DEC (1)

DAEA, DAEB, DAEC (1), DBEA, DBEB, DBEC (1), DCAB (1), DCAC (1), DCAE, DCBA (1), DCBC (1), DCBE, DEAB (1), DEAC (1), DEAE, DEBA (1), DEBC (1), DEBE

DAEB (1), DAEAC (1), DAEAE (2), DAEB (1), DAEBB (1), DAEBA (1), DAEBC (1), DAEBE, DBEAB (1), DBEAC (1), DBEAE, DBEBA (1), DBEBC (1), DBEBE (3), DCAEA, DCAEB, DCAEC (1), DCBEA, DCBEB, DCBEC (1), DEAEA (4), DEAEB (4), DEAEC (1), DEBBA (5), DEBEB (6), DEBEC (1)
Notes on the D-blocks

(1) A is a prefix of B, C. B is a prefix of C. The word cBA is a repetition, and also appears in cBC. The word CE is illegal as it appears in the context CEd3 $\Rightarrow$ 2ecd32ecd3. Also CD is illegal, leading to CEd3 $\Rightarrow$ 2cd32cd3. Thus CE is always followed by A or B, and hence db. The word EC therefore appears in context ECdb $\Rightarrow$ ecdbecdb.

(2) Contains AEAE.
(3) Contains BEBE.

(4) Contains EAEA.

(5) Contains cEBEA = c E db c E db.

(6) Contains EBEB.

Summary of useful D-blocks

DA, DB (1), DC (1), DE (1)

DAE (1), DBE (1), DCA, DCB (1), DEA, DEB (1)

DAEA, DAEB (1), DBEA, DBEB (1), DCAE (1), DCBE (1), DEAE (1), DEBE (1)

DAEBE (1), DBEAE (1), DCAEA, DCAEB (1), DCBEA, DCBEB (1)

DAEBEA, DCAEBE (1), DCBEAE (1),

DAEBEA (1), DCAEBEA,

DCAEBEAE (1)

Notes: (1) A combination (d-block other than A) D d3 will contain 2c d32c d3, a repetition.
Summary of remaining blocks

DA (1)
DCA (3)
DEA (2)
DAEA (4)
DBEA (6)
DCAEA (5)

α: DCBEA
β: DAEBEA
γ: DCAEBEA

Notes
(1) Appears as A DA D.
(2) After (1) is gone, appears as 2ecA DEA Dd3, a repetition.
(3) After (2) is gone, this block always appears in the context
   EA DCA Ddb ⊃ ec A D dbec A D db.
(4) After eliminations (1) - (3), this block appears only as EA DAEA Ddb, a repetition.
(5) Here DCAEA D3 ⊃ 2ec A d32ec A d3, a repeat.
(6) After (1) - (5) are eliminated, this block appears
only as
BEA DBEA D.

We are thus left with blocks $\alpha$, $\beta$, $\gamma$ with which to form a non-repetitive word of type $\omega$. However, it follows from our remarks in Chapter 1, in the first open problem, that if $v$ is a non-repetitive word concatenated from $\alpha$, $\beta$, $\gamma$, then $v$ must contain all of the three letter subwords $\alpha\beta\gamma$, $\alpha\gamma\beta$, $\beta\alpha\gamma$, $\beta\gamma\alpha$, $\gamma\alpha\beta$. We conclude that MAX.3 is not versatile.

MAX.4

Suppose that MAX.4 is versatile, and let $v$ be a non-repetitive word of type $\omega$ walkable on MAX.4. One checks that $\text{MAX.4} \setminus \{1\}$ may be reduced to the three element path, and is not versatile. Therefore $v$ must contain a 4. By analyzing the 1-blocks of MAX.4, we can conclude that MAX.4 is not versatile.

Level 1: 1-blocks:

12345 (1)
1234565 (2)
a: 1234562345
b: 123456345
12345632345 (3)
c: 1234563234565
d: 1234563234562345

**Notes on the 1-blocks**

(1) This block is a prefix of the others, and hence must be discarded.

(2) After (1) is gone, this block always appears in the context 5 1234565 123456, which is a repetition.

(3) This block is a prefix of c and d, and hence cannot be followed by them in a non-repetitive word. However, 2345 a 123456 is a repetition, as is 345 b 123456, so that the block must be discarded once (1) and (2) are gone.

We must include block c in v; otherwise block b can only occur in the context 345 b 123456, which is impossible. We are then left with only the two blocks a and d.

**Level 2: c-blocks:**

ca, cb, cd (1)

cab (2), cad (3), cba, cbd

cbab (2), cbad (3), cbda (3), cbdb (4)
Notes on the c-blocks

(1) The word 5cd is a repetition.

(2) Since ab123456 contains a repetition.

(3) The word da is illegal. Thus ad is illegal as ada is illegal and adb, adc contain repetitions.

(4) The word db is illegal as it appears in the context db123456 > 345b123456, which is a repetition.

Summary of useful c-blocks

α: ca
   cb (1)

β: cba

γ: cb (1)

(1) 345 cb c123456 = 345 c 123456 345 c 123456

The three blocks α, β, γ cannot be concatenated to form a non-repetitive word of type w, since αβγ contains a repetition. Thus MAX.4 is not versatile.

MAX.5

Suppose that MAX.5 is versatile, and let v be a
non-repetitive word of type \( \omega \) walkable on \( \text{MAX.5} \). One checks as in the previous case that 1 occurs in \( \nu \).

**Level 1: 1-blocks:**

123456 (1)
12345676 (2)

a: 123456756
b: 12345673456
1234567323456 (3)
c: 123456732345676
d: 1234567323456756

**Notes on the 1-blocks**

(1) This block is a prefix of the others, and is discarded.

(2) After (1) is gone this block always appears in context 6 12345676 1234567, a repetition.

(3) This block is a prefix of c and d, and hence cannot be followed by them in a non-repetitive word. However, 56 a 1234567 is a repetition, as is 3456 b 1234567, so that the block is discarded.

The word bac contains a repetition, so that we cannot concatenate \( \nu \) from a, b, c alone.
Level 2: d-blocks:

\[ da \ (1), \ db, \ dc \ (2) \]

\[ dba \ (1), \ dbc \]

\[ dbca, \ dbcb \]

\[ dbcab, \ dbcac \ (4), \ dbcba \ (1), \ dbcbc \ (3) \]

\[ dbcaba \ (1), \ dbcabc \ (5) \]

Notes on the d-blocks

(1) The word 56 a 1234567 is a repetition so that ba and da are illegal.

(2) Since 56dc is a repetition.

(3) Here bcbc is a repetition.

(4) Since dbcac 1234567 ≠ 56c1234567 56c1234567, a repetition.

(5) Here dbcabc 1234567 will contain the repetition 56bc1234567 56bc1234567.
Summary of useful d-blocks

\[\text{db}_a (1)\]

\[\text{dbc} (2)\]

\[\alpha: \text{dbca}\]

\[\beta: \text{dbcb}\]

\[\gamma: \text{dbcab}\]

(1) A prefix of the remaining blocks.

(2) A prefix to the blocks remaining after the elimination of (1).

The three blocks \(\alpha, \beta, \gamma\) cannot be concatenated to form a non-repetitive word of type \(\omega\), since \(\alpha\gamma\) contains a repetition. Thus \(\text{MAX.5}\) is not versatile.

\(\text{MAX.6}\)

Suppose that \(\text{MAX.6}\) is versatile, and let \(v\) be a non-repetitive word of type \(\omega\) walkable on \(\text{MAX.6}\). As in the previous two cases, \(v\) must contain a 1.

Level 1: 1-blocks:

\[123 (1)\]
Notes on the 1-blocks

(1) This block is a prefix of the others.

(2) After (1) is gone, this block is a prefix of the others.

(3) The words bf, cf, df, ef, af are illegal. This last, af, is illegal since fa is illegal and af123453 > 23454123453123454123453.

Suppose that c does not occur in v. Then v is formed from a, b, d, e, and b is a prefix of d, a prefix of e, a prefix of f. This is impossible by the Block Theorem.

To show that MAX.6 is not versatile, we consider blocks ending in c. Note that since cf contains a repetition

Level 2: reverse c-blocks:

ac (2), bc (1), dc, ec (2),
adc, bdc (1), edc (3),
badc, dadc (4), eadc (5),
abadc (6), dbadc (7), ebadc,
aebadc, bebadc (1), debadc (1)
baebadc (8), daebadc, eaebadc (5)
adaebadc (9), bdaebadc (1), edaebadc (3)

Notes on the reverse c-blocks

(1) The words be, bd, be, bf, de, df, ef contain repetitions.

(2) Here ac gives either ac 123453, a repetition, or ac a 12345 = ac a 12345, a repetition.

(3) Here ed leads to one of be, de, or ed, each repetition.

(4) Leads to bd or dad12345.

(5) As ac 12345 is a repetition.

(6) Since ab is repeated.

(7) Contains 312345312345.
(8) We get $3b12345$ or $4$ bae $4$ bad.
(9) Since ad is repeated.

**Summary of useful reverse c-blocks**

```
dc (1)
adc (2)
badc (3)
ebadc (4)
aebadc (2)
daebadc
```

(1) A suffix of the remaining blocks.
(2) As ca leads to $412345412345$.
(3) After (1) - (2) are gone, this block is a suffix of the remaining blocks.
(4) Appears in the context $4$badc ebadc d, a repetition.

MAX.6 is not versatile.

**MAX.7.**

Suppose that MAX.7 is versatile, and let $v$ be a non-repetitive word of type $w$ walkable on MAX.7. Again, MAX.7 \ (1) reduces to a three element path. The 1-blocks
of MAX.7 are
   a: 1232
   b: 123456
   c: 12345676
   d: 1234567656
Thus by the Block Theorem, MAX.7 is not versatile.

MAX.8

Suppose that MAX.8 is versatile, and let \( \nu \) be a non-repetitive word of type \( \omega \) walkable on MAX.8. We check that MAX.8 \( \setminus \{4\} \) reduces to a three element path. The 4-blocks of MAX.8 are
   a: 4123
   b: 45
   c: 4563
   d: 456323
Thus by the Block Theorem, MAX.8 is not versatile.

MAX.9

Suppose that MAX.9 is versatile, and let \( \nu \) be a non-repetitive word of type \( \omega \) walkable on MAX.9. We check that MAX.9 \( \setminus \{1\} \) is mimicked by MAX.7, which has been shown not to be versatile. One may therefore assume that
v contains a 1.

Level 1: 1-blocks:

12345 (1)

\[a:\] 1234562345
12345645 (2)

\[b:\] 1234564345

c: 123456434562345 (3)

d: 123456434562345645

e: 12345643456234564345

Notes on the 1-blocks

(1) This block is a prefix of the others.

(2) After (1) is gone, this block always appears in the context 45 12345645 123456, which is a repetition.

Level 2: a-blocks:

\[ab, ac, ad, ae\]

abc (1), abd (1), abe (1), acb (2), acd (1), ace (1), adb, adc, ade (1), aeb (3), aec, aed (4)

adbc (1), adbd (1), adbe (1), adcb, adcd (1),
adce (1), aecb, aecd (1), aece (1)
Notes on the a-blocks

(1) The words bc, bd, be, cd, ce, de are illegal.
(2) Since acb \Rightarrow 62345b 62345b.
(3) The word aeb1 contains 2345643451 2345643451.
(4) The word aed contains a repetition of
34512345643456234564.

Summary of useful a-blocks

ab (2)
ac (1)
ad (3)
ae (4)
adc (1)
a:  adb  
eaec (1)
\beta:  adcb 
aecb
(1) \( ca \ 123456 \succ 2345 \ 1234562345 \ 123456. \)

(2) Prefix of other blocks.

(3) After elimination (2), the a-block ad must occur in the context 5adae, which contains a repetition.

(4) Among the remaining blocks, ae appears either in the context aeae, or as b ae ad \( \succ (345 \ a \ 12345643456234564) \). The three blocks \( \alpha, \beta, \tau \) cannot be concatenated to form a non-repetitive word of type \( \omega \), since \( \alpha \beta \) contains a repetition. Thus MAX.9 is not versatile.

MAX.10

Suppose that MAX.10 is versatile, and let \( v \) be a non-repetitive word of type \( \omega \) walkable on MAX.10. One checks that vertex 4 must occur in \( v \). However, MAX.10 is versatile if and only if its reverse is. The 4-blocks of the reverse of MAX.10 are

a: 43
b: 432
c: 43215
d: 465

Invoking the Block Theorem, MAX.10 is not versatile.
MAX.11

One checks that 4 cannot be discarded from MAX.11. The 4-blocks of the reverse of MAX.11 are

a: 432
b: 4321
c: 43212
d: 45

This is impossible by the Block Theorem. Thus MAX.11 is not versatile.

MAX.12

Suppose that MAX.12 is versatile, and let v be a non-repetitive word of type ω walkable on MAX.12. One checks that 1 must appear in v. The 1-blocks of MAX.12 are

12
123
12345
a: 1234565
b: 1234562
c: 12345623
d: 1234562345
The block 12 is not useful, as it is a prefix of the other blocks. Similarly, the blocks 123, 12345 are eliminated from consideration. However, by the Block Theorem, the remaining blocks cannot be concatenated to form a non-repetitive word of type \( \omega \). Thus MAX.12 is not versatile.

**MAX.13**

Suppose MAX.13 is versatile for some \( q \). Let \( v \) be a non-repetitive word of type \( \omega \) walkable on MAX.13. One checks that we may assume that \( v \) contains a 1. The 1-blocks of MAX.13 are:

- 12
- 123
- 1234

... 

- 1234...(q-1)
- 123...q2
- 123...q23
- 123...q234
123...q234...(q-1)

We see that the block 12 is a prefix of all the other blocks, and hence cannot appear in \( v \). Again, the block 123 is a prefix of all the other blocks excluding 12, and hence can never be used in \( v \). Continuing in this way, we can eliminate all the blocks in order, showing that none of them can be used in \( v \), which is a contradiction. Thus \( \text{MAX.13} \) is not versatile for any \( q \).

**MAX.14**

Suppose that \( \text{MAX.14} \) is versatile, and let \( v \) be a non-repetitive word of type \( \omega \) walkable on \( \text{MAX.14} \). One checks that \( v \) may be assumed to commence with 1. The 1-blocks of \( \text{MAX.14} \) are

- \( a: 1232 \)
- \( b: 1234 \)
- \( c: 123456 \)
- \( d: 12345676 \)

By the Block Theorem, these blocks cannot be concatenated to form a non-repetitive word of type \( \omega \). Thus \( \text{MAX.14} \) is
not versatile.

MAX.15

Suppose that MAX.15 is versatile, and let v be a non-repetitive word of type ω walkable on MAX.15. One checks that v may be assumed to contain a 5. The 5-blocks of MAX.15 are

a: 56
b: 51234
c: 512343234

Since bc contains a repetition, these blocks cannot be concatenated to form a non-repetitive word of type ω. Thus MAX.15 is not versatile.

MAX.16

Suppose that MAX.16 is versatile, and let v be a non-repetitive word of type ω walkable on MAX.16. One checks that v may be assumed to contain a 3. The 3-blocks of MAX.16 are

a: 312
b: 342
Since abc contains a repetition, these blocks cannot be concatenated to form a non-repetitive word of type \( \omega \). Thus MAX.16 is not versatile.

**MAX.17**

Suppose that MAX.17 is versatile, and let \( v \) be a non-repetitive word of type \( \omega \) walkable on MAX.17. One checks that \( v \) may be assumed to contain a 3.

**Level 1: 3-blocks:**

<table>
<thead>
<tr>
<th>Block</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>32</td>
</tr>
<tr>
<td>b</td>
<td>312</td>
</tr>
<tr>
<td>c</td>
<td>3412</td>
</tr>
<tr>
<td>d</td>
<td>345</td>
</tr>
<tr>
<td>e</td>
<td>34512</td>
</tr>
</tbody>
</table>

One checks that \( v \) may be assumed to contain a d.

**Level 2: d-blocks:**

- da, db, dc, de (1)
- dab, dac, dae, dba (2), dbc (3), dbe (5), dca
(2), dcb (4), dce (5)

daba (2), dabc (3), dabe (5), daca (2), dacb (4),
dace (5), daea (2), daeb (4), daec (3)

Notes on the d-blocks

(1) The word de contains a repetition.
(2) Since 2a3 is a repetition.
(3) Here 12c will either appear in context
    12c34, a repetition,
    12cb3 > 12b3, a repetition,
    or 12ca3 > 2a3, a repetition.
(4) As noted above, 12b3 is a repetition.
(5) Here 12e will appear in one of the following contexts:
    12 e 345 , a repetition,
    12 e c > 12 c, leading to a repetition as in (3)
above,
    12 e a > 2 a, leading to repetition as in (2)
or 12 e b > 12 b, leading to the repetition of (4).

Summary of useful d-blocks

da (1)
(1) 2dad3 is a repetition.

(2) After the elimination (1), this block appears in context
12 db d3, a repetition

Thus if MAX.17 is versatile, a non-repetitive word can be composed of blocks A,B,C,D. However this would imply that the following blocks could be concatenated to form a non-repetitive word of type w:

A': 12d34
B': 12da3
C': 12da34
D': 12da345

As this is impossible by the Block Theorem, MAX.17 is not versatile.
Suppose that MAX.18 is versatile, and let \( v \) be a non-repetitive word of type \( \omega \) walkable on MAX.18. One checks that \( v \) may be assumed to contain a 1.

**Level 1: 1-blocks:**

1. 12345 (1)
2. 123456 (2)
3. \( a: \) 123456
4. \( b: \) 123456345
   - 12345632345 (3)
5. \( c: \) 123456323456
6. \( d: \) 1234563234565

**Notes on the 1-blocks**

(1) This block is a prefix of the others.

(2) After (1) is gone, this block is a prefix of the remaining blocks.

(3) This block is a prefix of \( c \) and \( d \), and hence cannot be followed by them in a non-repetitive word. However, \( 5 \) a 123456 is a repetition, as is 345 b 123456, so that this block is also discarded.

One checks that \( v \) may be assumed to contain a 1.
Level 2: c-blocks:

ca, cb, cd (1)

cab, cad (2), cba (1), cbd

caba (1), cabd, cbda (1), cbdb

cabda (1), cabdb, cbdba (1), cbdbd (3)
cabdba (1), cabdbd (3)

Notes on the c-blocks

(1) The word 5a leads to a repetition 5 1234565 123456, so that words ba, da are not useful. The word cd contains a repetition.

(2) Since da is illegal, so is ad, which must occur in the context adx, x = b or c, hence ad1234563 > 23456512345632345651234563.

(3) Repeats bd.

Summary of useful c-blocks

cia (2)
cb (3)
cab (4)
cbd (1)
cabd (1)
cbdb
cabdb

(1) As $5dc > 5 \ 1234563234565 \ 123456323456$.

(2) For, 5 cac $123456 > 5c1234565c123456$.

(3) After (1), (2) are gone, cb always appears in context b cb c.

(4) Here db cab c 1 $\geq 234565 b c \ 1234565 b c 1$.

We are left with only two blocks. Thus MAX.18 is not versatile.

**MAX.19**

Suppose that MAX.19 is versatile, and let $v$ be a non-repetitive word of type $\omega$ walkable on MAX.19. One checks that $v$ may be assumed to contain a 1.
Level 1: 1-blocks:

1234 (1)
12345 (2)
a: 12345234
b: 1234534
123453234 (3)
c: 1234532345
d: 1234532345234

Notes on the 1-blocks
(1) This block is a prefix of the others.
(2) After (1) is gone, this block is a prefix of the remainder.
(3) This block is a prefix of c and d, and hence cannot be followed by them in a non-repetitive word. However, 234 a 12345 is a repetition, as is 34 b 12345, so that the block is discarded.

One checks that v may be assumed to contain a c.

Level 2: c-blocks:

ca, cb, cd (1)

cab (2), cad (3), cba (1), cbd
Notes on the c-blocks

(1) The word cd contains a repetition.
(2) Since $34 \ b \ 12345$ is a repetition.
(3) Here ad leads to ad123453, a repetition, or adb12345, a repetition. (See (2).)
(4) Since $234 \ a \ 12345$ is a repetition.

Summary of useful c-blocks

- cb (1)
  - $\alpha$: ca
  - $\beta$: cba
  - $\gamma$: cbd

(1) As $34 \ cb \ c \ 12345$ is a repetition.

The three blocks $\alpha$, $\beta$, $\gamma$ cannot be concatenated to form a non-repetitive word of type $\omega$, since $\beta\alpha\gamma$ contains a repetition. Thus MAX.19 is not versatile.
Suppose that MAX.20 is versatile, and let \( v \) be a non-repetitive word of type \( \omega \) walkable on MAX.20. One checks that \( v \) may be assumed to contain a 6, since \( \text{MAX.20} \setminus \{6\} \) is mimicked by MAX.19.

**Level 1: 6-blocks**

- 612345 (1)
- 61234532345 (2)
- a: 612345323412345
- b: 6123453412345
- c: 612345341234532345
- d: 6123453412345323412345
- e: 61234534123453234123453412345

**Notes on 6-blocks**

1. This block is a prefix of the others.
2. After elimination (1), this block always occurs in the context 23456 12345323456 1234563, which is a repetition.

One checks that \( v \) may be assumed to contain an a.

**Level 2: a-blocks:**

- ab (1), ac, ad (2), ae
acb, acd (2), ace (2), aed (2), aec, aeb (2)

aecb, aecd (2), aec (2), aebc (2), aebd (2),
aebe (2)

aecbc (2), aecbd (2), aecbe (2)

**Notes on the a-blocks**

(1) The word ab always occurs in the context
ab 612345 > 3412345 612345 3412345 612345, a repetition.

(2) The following words contain repetitions:
   da6, ada6, ad61234534
   (Thus ad is illegal.)
   bc, bd, be, cd, ce, de
   eb6, aed.

**Summary of useful a-blocks**

ac (1)
B: ae
A: acb
C: aec
D: aecb
As 5a5e is illegal.

By the Block Theorem, MAX.20 is not versatile.

MAX.21

Suppose that MAX.21 is versatile, and let \( v \) be a non-repetitive word of type \( \omega \) walkable on MAX.21. One checks that \( v \) may be assumed to contain a 1. The 1-blocks of MAX.21 are

- 123 (1)
- 1234 (2)
- a: 12345
- b: 12343
- c: 1234323
- d: 12343234
- 123432345 (3)

Notes: Blocks (1) and (2) are eliminated as prefixes.

Block (3) cannot be preceded by any block but a, and thus must appear in one of two contexts:

- a 123432345 12345 1 \( \Rightarrow \) 234512343 234512343
- a 123432345 12345 1 \( \Rightarrow \) 23451 23451.

By the Block Theorem, the remaining blocks cannot be
concatenated to form a non-repetitive word of type \( \omega \). Thus MAX.21 is not versatile.

**MAX.22**

Suppose that MAX.22 is versatile, and let \( v \) be a non-repetitive word of type \( \omega \) walkable on MAX.22. One checks that \( v \) may be assumed to contain a 1. The 1-blocks of MAX.22 are

- 12
- 123
- 1234
- \( a: \ 123456 \)
- \( b: \ 123452 \)
- \( c: \ 1234523 \)
- \( d: \ 12345234 \)

The first three blocks are eliminated in turn, as prefixes. By the Block Theorem, the remaining blocks cannot be concatenated to form a non-repetitive word of type \( \omega \). Thus MAX.22 is not versatile.
Suppose that MAX.23 is versatile, and let \( v \) be a non-repetitive word of type \( \omega \) walkable on MAX.23. One checks that \( v \) may be assumed to contain a 1. The 1-blocks of MAX.23 are

12, which is discarded

\[ \begin{align*}
    a: & \quad 1232 \\
    b: & \quad 12345 \\
    c: & \quad 123456 \\
    d: & \quad 1234565 
\end{align*} \]

By the Block Theorem, these blocks cannot be concatenated to form a non-repetitive word of type \( \omega \). Thus MAX.23 is not versatile.

MAX.24

Suppose that MAX.24 is versatile, and let \( v \) be a non-repetitive word of type \( \omega \) walkable on MAX.24. One checks that \( v \) may be assumed to contain a 1. The 1-blocks of MAX.24 are

12, which is discarded

\[ \begin{align*}
    a: & \quad 1232 
\end{align*} \]
b: 1234
c: 12345
d: 123456

By the Block Theorem, these blocks cannot be concatenated to form a non-repetitive word of type \( w \). Thus MAX.24 is not versatile.

**MAX.25**

Suppose that MAX.25 is versatile, and let \( v \) be a non-repetitive word of type \( w \) walkable on MAX.25. One checks that \( v \) may be assumed to contain a 2. By analyzing the 2-blocks of MAX.25, we can conclude that MAX.25 is not versatile.

**Level 1: 2-blocks:**

a: 21
b: 234
c: 23454
d: 23451

One checks that \( v \) may be assumed to contain an a.
Level 2: a-blocks:

ab, ac, ad

abc (1), abd (1), acb, acd, adb, adc (2)

acbc (1), acbd (2), acdb, acdc, adbc (1),

adbd (2)

acdbc (1), acdbd (1), acdcb, acdcd (3)

acdcdbc (1), acdcbd (1)

Notes on the a-blocks

(1) The word b is a prefix of c and d. (2) This block followed by 3 contains 2a3 2a3.
(2) Here adc > 1234512345.
(3) The word cd repeats.

Summary of useful a-blocks

ab (2)

F: ac

ad (1)
A: adb
    acd (1)
B: acb
C: acdb
D: acdc
E: acdcb

Notes

(1) The block da2 $\Rightarrow$ 1212.
(2) A prefix of the other blocks.

We are left with five a-blocks:

After one more level of blocks, we are done.

Level 3: A-blocks:

AB, AC, AD, AE, AF (1)

ABC (2), ABD (2), ABE (2), ACB, ACD (3), ACE (3), ADB,
    ADC (6),
    ADE (4), AEB (5), AEC, AED (7)

ACBC (2), ACBD (2), ACBE (2), ADBC (2), ADBD (2), ADBE
    (2), AECB,
AECD (3), AECE (3),

AECBC (2), AECBD (2), AECBE (2).

Notes on the A-blocks

(1) A prefix of the other blocks.
(2) B is a prefix of C, D, E.
(3) C is a prefix of D, E.
(4) D prefix of E.
(5) Here AEBa \Rightarrow cb acb a.
(6) As DC contains dcac2 \Rightarrow 1c21c2.
(7) As AED \Rightarrow b acdcb acdc.

Summary of useful A-blocks

AB (1)
AC (2)
AD (6)
AE
ACB (5)
ADB (4)
AEC (3)
AECD
(2) Once (1) is gone, this block is a prefix.

(3) Leads to CA a → db a

(4) DB → dcac2 → 1c2 l2c2.

(5) ACB → db acdb ac.

(6) After (1) – (5) are gone, this block is prefix of the remainder.

We are thus left with only two blocks. These blocks cannot be concatenated to form a non-repetitive word of type ω.

MAX.26

Suppose that MAX.26 is versatile, and let v be a non-repetitive word of type ω walkable on MAX.26. One checks that v may be assumed to contain a 6. The 6-blocks of MAX.26 are

67, which is discarded

a: 6787

b: 67812345

c: 678123454345

d: 6781234543452345

By the Block Theorem, these blocks cannot be concatenated to form a non-repetitive word of type ω. Thus MAX.26 is
not versatile.

We have now established that none of the digraphs of MAX are versatile.
Bibliography


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Appendix I: Digraphs of MIN
Appendix II: Digraphs of MAX
\text{MAX.16}

\text{MAX.17}

\text{MAX.18}
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