Counting endomorphisms of crown-like orders

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Abstract

The authors introduce the notion of crown-like orders and introduce powerful tools for counting the endomorphisms of orders of this type.

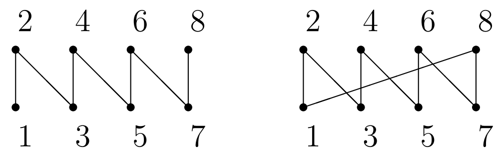


Figure 1: The fence and crown of order 4

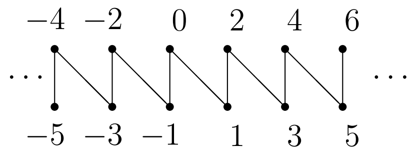


Figure 2: The infinite fence $\mathcal{F}$

1 Introduction

Let n be a natural number. By the **fence of order** n we mean the order $\mathcal{F}_n$ on $\{1, 2, \ldots, 2n\}$ where the odd elements are minimal, the even elements are maximal, and elements i and j are comparable exactly when i and j differ by 1. The **crown of order** n is the order $\mathcal{C}_n$ on $\{1, 2, \ldots, 2n\}$ where the odd elements are minimal, the even elements are maximal, and elements i and j are comparable exactly when i and j differ by 1 modulo $2n$. (See Figure 1.)

Crowns and fences are two types of orders for which the number of endomorphisms is known exactly [1, 2]. Duffus et al. assert that the methods of [2] should extend to give asymptotic estimates for the number of endomorphisms of the orders depicted in Figure 3.

One sees that the orders in Figure 3 are 'fence-like' in their repetition of a certain basic unit. In this paper we propose a definition for fence-like and crown-like orders which will include the orders of Figure 3 as special cases. We also introduce a powerful method of exactly counting the endomorphisms of crown-like orders.

2 Generalized Crowns

In [2] it turns out to be useful to consider the **infinite fence**. This is the order $\mathcal{F}$ on $\mathbb{Z}$ where the odd elements are minimal, the even elements are maximal, and elements i and j are comparable exactly when i and j differ by 1. (See Figure 2.) This order has the shift map $\sigma_2 : \mathbb{Z} \to \mathbb{Z}$ given by $\sigma_2(i) = i + 2$ as an automorphism. The restriction of $\mathcal{F}$ to $\{1, 2, \ldots, 2n\}$ is $\mathcal{F}_n$, while $\mathcal{C}_n$ arises from $\mathcal{F}$ by identifying elements which are congruent modulo $2n$.

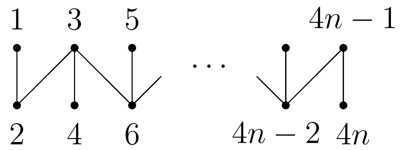
Say that order F on $\mathbb{Z}$ is a **generalized infinite fence** if

- the diagram of F is connected
- at most finitely many elements are comparable to any element of F
- for some natural number s, F has the shift map $\sigma_s : \mathbb{Z} \to \mathbb{Z}$ given by $\sigma_s(i) = i + s$ as an automorphism.

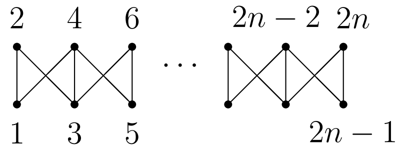
Let F on $\mathbb{Z}$ be a **generalized infinite fence** with s_0 a natural number such that σ_{s_0} is an automorphism of F. Let k be a natural number such that no element of F greater than ks_0 is comparable to an element of $\{1, 2, \ldots, s_0\}$. We call the order P induced on $\{1, 2, \ldots, ks_0\}$ by F the **base unit** of F. Every comparability of F is recoverable from its base unit. Suppose that $i, j \in \mathbb{Z}$, with $i < j$. Then $i <_F j$ if and only if $i' <_P j'$, where $1 \leq i' \leq s_0$, $i' \equiv i \bmod s_0$ and $j' - i' = j - i$. In the case where i' and j' are comparable, it will follow that $j' \leq ks_0$, so that $i', j' \in P$. One similarly determines whether $i >_F j$. It therefore makes sense to call F **the infinite fence with base unit** P **and period** s_0. We write $F = F(P, s_0)$.

Suppose that $F = F(P, s_0)$ is given, with $|P| = ks_0$. There are infinitely many other ways of choosing s_0 and k for F. Suppose t is an integer. Let $\hat{s}_0 = s_0 + ts_0$, $\hat{k} = k + t$. Then $\sigma_{\hat{s}_0}$ is an automorphism of F and no element greater than $\hat{k}_0$ is comparable to any element of $\{1, 2, \ldots, \hat{k}s_0\}$. Choosing $t = k - 2$, we get $\hat{k}s_0 = (k + t)s_0 = (2k - 2)s_0 = 2(s_0 + ts_0) = 2\hat{s}_0$. Choosing s_0 appropriately, we can thus always pick $k = 2$. We shall do this in the remainder of this paper.

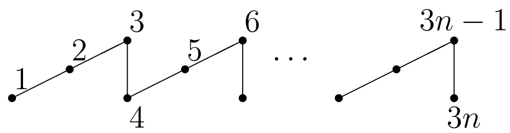
Given a generalized infinite fence $F = F(P, s_0)$ and a natural number n, the **generalized fence of order** n **with base unit** P**, period** s_0 is the order $F_n(P, s_0)$ which is the restriction of F to $\{1, 2, \ldots, ns_0\}$. The **generalized crown of order** n **with base unit** P**, period** s_0 is the order $C_n(P, s_0)$ on $\{1, 2, \ldots, ns_0\}$ which is obtained from F by identifying elements which are equivalent modulo ns_0.



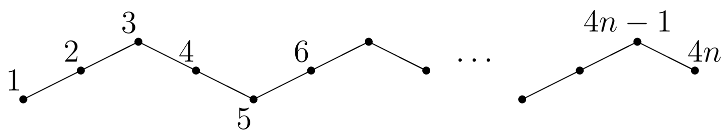
Fence $\mathcal{A}_n$



Fence $\mathcal{B}_n$



Fence $\mathcal{D}_n$



Fence $\mathcal{E}_n$

Figure 3: Various generalized fences of order n

Example 2.1 Consider the ordinary infinite fence $\mathcal{F}$ depicted in Figure 2. Here $s_0 = 2$ and P is the order induced by $\mathcal{F}$ on $\{1, 2, 3, 4\}$. Then $\mathcal{F} = F(P, 2), \mathcal{F}_n = F_n(P, 2), \mathcal{C}_n = C_n(P, 2)$. For the orders in Figure 3, we could choose $s_0 = 4$, 2, 3 and 4 respectively.

3 Counting Endomorphisms

Let a generalized infinite fence $F = F(P, s_0)$, and a natural number $n \geq 2$ be given. We will abbreviate σ_{s_0} by σ, $C_n(P, s_0)$ by C_n and $F_n(P, s_0)$ by F_n.

Remark 3.1 The crown C_n is obtained by identifying elements of F which are equivalent modulo ns_0. Let $i, j \in \{1, 2, \ldots, ns_0\}$ be given, with $i < j$. Suppose that $i <_{C_n} j$. Then $i' <_F j'$ some $i', j' \equiv i, j$ modulo ns_0. Since the base unit of F is P, of size $2s_0$, we can specify that the difference between i' and j' is less than $2s_0$. Choosing the lesser of i', j' to lie in $\{1, 2, \ldots, ns_0\}$, we can pick i', j' to lie in $\{1, 2, \ldots, ns_0 + ks_0 = (n + 2)s_0\}$. A similar result holds if $i >_{C_n} j$. Thus every comparison in C_n will be realized if we identify elements of F_{n+2} which are congruent modulo ns_0, and C_n is obtained from identifying the first $2s_0$ and the last $2s_0$ elements of F_{n+2}.

The following are immediate:

Lemma 3.2 *Endomorphisms of C_n are in 1-1 correspondence with those homomorphisms g of F_{n+2} into C_n for which*

$$g(i) = g(i + ns_0),\ i = 1, 2, \ldots, 2s_0.$$

Lemma 3.3 *Homomorphisms g of F_{n+2} into C_n for which $g(i) = g(i + ns_0)$, $i = 1, 2, \ldots, 2s_0$ are in 1-1 correspondence with those homomorphisms f of F_{n+2} into F for which*

$$\begin{aligned} f(1) &\in \{1, 2, \ldots, ns_0\} \text{ and} \\ f(i) &\equiv f(i + ns_0) \bmod ns_0,\ i = 1, 2, \ldots, 2s_0. \end{aligned}$$

Note that the homomorphisms of F_{n+2} into F for which $f(1) \in \{1, 2, \ldots, ns_0\}$ are in n to 1 correspondence with homomorphisms f of F_{n+2} into F for which $f(1) \in \{1, 2, \ldots, s_0\}$. This gives the following:

Lemma 3.4 *Endomorphisms of C_n are in n to 1 correspondence with those homomorphisms of F_{n+2} into F for which*

$$\begin{aligned} f(1) &\in \{1,2,\ldots,s_0\} \text{ and} & (1) \\ f(i) &\equiv f(i+ns_0) \bmod ns_0,\ i=1,2,\ldots,2s_0. & (2) \end{aligned}$$

Let us consider homomorphisms from F_{n+2} into F. Note that $F_{n+2} = \cup_{i=0}^{n} \sigma^i(P)$. Let a homomorphism f from F_{n+2} to F be given. Homomorphism f corresponds to a sequence of homomorphisms $\prod_{i=0}^{n}\{f_i : P \to F\}$ where

$$f_i(p) = f \circ \sigma^i(p), p \in P. \quad (3)$$

In fact, $f = \cup_{i=0}^{n} f_i \circ \sigma^{-i}$.

Definition 3.5 Define operators **head** and **tail** from F^P to F^{F_1} by

$$\begin{aligned} \mathbf{head}(g) &= g_{|F_1} \\ \mathbf{tail}(g) &= g_{|\sigma(F_1)} \circ \sigma \end{aligned}$$

If f_i are defined as in (3), then we must have

$$\mathbf{tail}(f_i) = \mathbf{head}(f_{i+1}) \text{ for } i = 0,1,\ldots,n-1. \quad (4)$$

Conversely, if a sequence of homomorphisms $\prod_{i=0}^{n}\{f_i : P \to F\}$ is given satisfying (4), then $f = \cup_{i=0}^{n} f_i \circ \sigma^{-i}$ is a well-defined homomorphism from F_{n+2} to F. It follows that homomorphisms from F_{n+2} to F are in 1-1 correspondence with sequences of homomorphisms $\prod_{i=0}^{n}\{f_i : P \to F\}$ satifying (4).

If $f : F_{n+2} \to F$ also satisfies (1),(2) , then f corresponds to a sequence of homomorphisms $\prod_{i=0}^{n}\{f_i : P \to F\}$ where $f_0 \equiv f_n$ modulo ns_0, and thus to a sequence $\prod_{i=0}^{n-1}\{f_i : P \to F\}$ where

$$\begin{aligned} \mathbf{tail}(f_i) &= \mathbf{head}(f_{i+1}) \text{ for } i = 0,1,\ldots,n-2, \\ \mathbf{tail}(f_{n-1}) &\equiv \mathbf{head}(f_0) \text{ modulo } ns_0. \end{aligned}$$

Definition 3.6 Let $\simeq$ be the equivalence relation on homomorphisms from F_1 to F given by

$$h \simeq g \text{ if and only if } h = \sigma^k \circ g, \text{ some } k \in \mathbb{Z}.$$

Given any homomorphism g from F_1 to F, let $\bar{g}$ be the unique homomorphism such that $g \simeq \bar{g}$ and $\bar{g}(1) \in \{1,2,\ldots,s_0\}$.

Remark 3.7 Suppose that $h \simeq g$. The k for which $h = \sigma^k \circ g$ is unique. Thus if $h(1) \equiv g(1)$ modulo ns_0 then $h \equiv g$ modulo ns_0.

Definition 3.8 The **type** of a homomorphism $f : P \to F$ is the ordered pair

$$\textbf{type } f = \langle\, \overline{\textbf{head}(f)},\ \overline{\textbf{tail}(f)}\rangle.$$

If $f : F_{n+2} \to F$ satisfies (1),(2) , then f corresponds to a sequence of homomorphisms $\prod_{i=0}^{n-1}\{f_i : P \to F\}$ where

$$\pi_2(\textbf{ type } f_i) = \pi_1(\textbf{ type } f_j), \text{ for } j \equiv i+1 \text{ modulo } n. \tag{5}$$

Here π_1, π_2 are the projections.

This has a partial converse; suppose we are given a sequence of homomorphisms $\prod_{i=0}^{n-1}\{f_i : P \to F\}$ satisfying (5). Define $f_n = f_0$. Let $\hat{f}_0 = \bar{f}_0$. Suppose that homomorphisms $\hat{f}_i$ have been given for $i < j \le n$ so that

$$\begin{aligned} \textbf{type } \hat{f}_i &= \textbf{type} f_i, \\ \textbf{tail } \hat{f}_{i-1} &= \textbf{head} \hat{f}_i \text{ for } i > 1. \end{aligned}$$

Since **type** $\hat{f}_{j-1} =$ **type** f_{j-1} and $\pi_2($ **type** $f_{j-1}) = \pi_1($ **type** $f_j)$, we have **tail** $\hat{f}_{j-1} \simeq$ **tail** $f_{j-1} \simeq$ **head** f_j. Choose k so that **tail** $\hat{f}_{j-1} = \sigma^k \circ$ **head** f_j. Define $\hat{f}_j = \sigma^k \circ f_j$. Then **type** $\hat{f}_j =$ **type** f_j, while **tail** $\hat{f}_{j-1} = \sigma^k \circ$ **head** $f_j =$ **head** $\sigma^k \circ f_j =$ **head** $\hat{f}_j$.

This is the unique way to construct homomorphisms $\hat{f}_i$ such that

$$\begin{aligned} \hat{f}_0(1) &\in \{1, 2, \ldots, s_0\} \text{ and} \\ \textbf{type } \hat{f}_i &= \textbf{type } f_i, \\ \textbf{tail } \hat{f}_{i-1} &= \textbf{head } \hat{f}_i \text{ for } 1 \le i \le n. \end{aligned}$$

There is no flexibility in this construction. Thus, while **type** $\hat{f}_n =$ **type** $\hat{f}_0$, so that **tail** $\hat{f}_{n-1} =$ **head** $\hat{f}_n \simeq$ **head** $\hat{f}_0$, we cannot guarantee that $\hat{f}_n \equiv \hat{f}_0$ modulo ns_0, which would imply that **tail** $\hat{f}_{n-1} \equiv$ **head** $\hat{f}_0$ modulo ns_0.

The homomorphism $f = \cup_{i=0}^{n} \hat{f}_i$ is a well-defined homomorphism from F_{n+2} to F, but may not satisfy (1),(2). If, however, $\hat{f}_n(1) \equiv \hat{f}_0(1)$ modulo ns_0, then we have not just **head** $\hat{f}_n \simeq$ **head** $\hat{f}_0$, but **head** $\hat{f}_n \equiv$ **head** $\hat{f}_0$ modulo ns_0, so that (1),(2) are satisfied.

Suppose that $r \in \mathbb{N}$ and $g : F_r \to F$ is a homomorphism. Assign a weight to g by

$$w(g) = g(s_0(r-1)+1) - g(1)$$

If we have homomorphisms $f_i : P = F_2 \to F, f : F_{n+2} \to F$ satisfying $\hat{f}_i(p) = f \circ \sigma^i(p), p \in P$, or equivalently, $f = \cup_{i=0}^n \hat{f}_i \circ \sigma^{-i}$, then

$$\begin{aligned}
\sum_{i=0}^{n} w(\hat{f}_i) &= \sum_{i=0}^{n} \hat{f}_i(s_0+1) - \hat{f}_i(1) \\
&= \sum_{i=0}^{n} f((i+1)s_0+1) - f(is_0+1) \\
&= f((n+1)s_0+1) - f(is_0+1) \\
&= w(f).
\end{aligned}$$

In this language, our previous discussion noted that $f : F_{n+2} \to F$ satisfies (1),(2) if and only if

$$\begin{aligned}
\sum_{i=0}^{n-1} w(\hat{f}_i) &= \hat{f}_{n-1}(s_0+1) - \hat{f}_0(1) \\
&= \hat{f}_n(1) - \hat{f}_0(1) \\
&\equiv 0 \text{ modulo } ns_0
\end{aligned}$$

Definition 3.9 There are finitely many homomorphisms h of P into F for which $h(1) \in \{1, 2, \ldots, s_0\}$. This is because $h(i)$ and $h(i+1)$ can differ by at most $m-1$, so that the range of h is restricted to $[1-(m-1)^2, s_0+(m-1)^2]$. Say that there are r possible equivalence classes for **head** (h), **tail** (h), labelled $1, 2, \ldots, r$. This gives us a natural labelling of homomorphism types:

type $h \in (i, j)$ if and only if **head**$(h) \in i$, **tail**$(h) \in j$.

For $i, j \in \{1, 2, \ldots, r\}$, let G_{ij} be the set containing those homomorphisms $h : P \to F$ for which $h(1) \in \{1, 2, \ldots, s_0\}$ and **type** $h = (i, j)$.

Let $\mathcal{G}_n$ be the set of homomorphisms f from F_{n+2} to F for which (1),(2) hold. Let $G_n = \cup \prod_{k=0}^{n-1} G_{i_k j_k}$ where the union is taken over sequences $\prod_{k=0}^{n-1}(i_k, j_k)$ with $i_k = j_r$ when $r \equiv k+1$ modulo n. Assign a weight to elements of G_n by $w(\prod_{k=0}^{n-1} f_i) = \sum_{i=0}^{n-1} w(f_i)$. Let $\mathcal{S}_n = \{x \in G_n : w(x) \equiv 0 \text{ modulo } ns_0\}$.

With this notation, our previous discussion is summarized by a theorem:

Theorem 3.10 *The map*

$$f \to \prod_{i=0}^{n-1} \{f_i\} \text{ where the } f_i \text{ are given by (3)}$$

is a bijection between $\mathcal{G}_n$ *and* $\mathcal{S}_n$.

Definition 3.11 If S is a set of homomorphisms, denote by Φ_S the generating homomorphism of S with respect to w:

$$\Phi_S(x) = \sum_{h \in S} x^{w(h)}$$

If $B = [b_{ij}]_{r \times r}$, then the ij^{th} entry of B^n is $\sum \prod_{k=0}^{n-1} b_{i_k j_k}$ where the sum is taken over sequences $\prod_{k=0}^{n-1}(i_k, j_k)$ with

$$i_0 = i,\ j_{n-1} = j \text{ and } j_k = i_{k+1} \text{ for } 0 \leq k \leq r-2.$$

If $i = j$, then we also have $j_{n-1} = i_0$.

These are just the restrictions on the indices of G_n. Let $A = [\Phi_{\mathcal{G}_{ij}}]_{r \times r}$. The notation $[y^n]h(y)$ refers to the coefficient of y^n in a series expansion of h.

Then

$$\begin{aligned}
\phi_{G_n} &= \phi_{\cup \prod_{k=0}^{n-1} G_{i_k j_k}} \\
&= \sum \prod_{k=0}^{n-1} \phi_{G_{i_k j_k}} \\
&= \text{trace}(A^n) \\
&= \text{trace}([y^n] \sum_{k \geq 0} (yA)^k) \\
&= [y^n]\text{trace}\left((I - yA)^{-1}\right)
\end{aligned}$$

Thus

$$|\mathcal{S}_n| = \sum_{t \equiv 0 \text{ modulo } ns_0} [x^t y^n] \text{ trace}\left((I - yA)^{-1}\right).$$

Theorem 3.12 *The number of endomorphisms of* C_n *is*

$$n \sum_{t \equiv 0 \textit{ modulo } ns_0} [x^t y^n] \textit{ trace}\left((I - yA)^{-1}\right).$$

4 Examples

4.1 Ordinary Crowns

Consider ordinary crowns $\mathcal{C}_n$. Order $\mathcal{F}_1$ is the 2-element chain with $1 <_{\mathcal{F}_1} 2$. If $f : \mathcal{F}_1 \to \mathcal{F}$ is a homomorphism with $f(1) \in \{1, 2\}$, we have 4 possibilities:

$$\begin{array}{ll}
(1) & f(1)=1, \quad f(2)=2; \\
(2) & f(1)=1, \quad f(2)=0; \\
(3) & f(1)=1, \quad f(2)=1; \\
(4) & f(1)=2, \quad f(2)=2.
\end{array}$$

If h is a homomorphism from $\mathcal{F}_m$ to $\mathcal{F}$, there are thus 4 possible equivalence classes for **head** (h), **tail** (h). To find the matrix A of Theorem 3.12 we need to find the G_{ij}. As an example, consider the set G_{12}. This set consists of two homomorphisms from $\mathcal{F}_2$ to $\mathcal{F}$, namely g_1 and g_2 where

$$\begin{array}{l}
g_1(1)=1, \quad g_1(2)=2, \quad g_1(3)=1, \quad g_1(4)=0; \\
g_2(1)=1, \quad g_2(2)=2, \quad g_2(3)=3, \quad g_2(4)=2.
\end{array}$$

We see that

$$\begin{array}{l}
w(g_1) = g_1(3) - g_1(1) = 1 - 1 = 0 \\
w(g_2) = g_2(3) - g_2(1) = 3 - 1 = 2.
\end{array}$$

Thus $\phi_{G_{12}} = 1 + x^2$. In this way we determine

$$A = \begin{bmatrix} 1+x^2 & 1+x^2 & 1+x^2 & x \\ x^{-2}+1 & x^{-2}+1 & x^{-2}+1 & x^{-1} \\ 1 & 1 & 1 & 0 \\ x^{-1}+x & x^{-1}+x & x^{-1}+x & 1 \end{bmatrix}.$$

Suppose that h is a homomorphism from $\mathcal{F}_{n+2}$ to $\mathcal{F}$. It must be the case that $|h(i)-h(i+1)| \le 1$ for each i. Thus $w(h) \equiv 0$ modulo $2n$ if and only if $w(h) = 0$ or $w(h) = \pm 2n$. The latter two weights occur exactly in the case when h corresponds to one of the $2n$ automorphisms of C_n.

For ease of computation, we replace A with $M = x^2 A$:

$$M = \begin{bmatrix} x^2+x^4 & x^2+x^4 & x^2+x^4 & x^3 \\ x^2+1 & x^2+1 & x^2+1 & x \\ x^2 & x^2 & x^2 & 0 \\ x^3+x & x^3+x & x^3+x & x^2 \end{bmatrix}.$$

Let c_n be the number of endomorphisms of a crown on $2n$ elements. Then

$$c_n = 2n + n[x^{2n}y^n]\mathrm{trace}\,(I-yM)^{-1}$$

and

$$\mathrm{trace}\,(I-yM)^{-1} = \frac{4-12yx^2+2y^2x^4-3y-3x^4y}{1-4yx^2+y^2x^4-y-x^4y}.$$

This expression is even in x, so we have

$$\begin{aligned} c_n &= 2n + n[x^ny^n]\frac{4-12xy+2x^2y^2-3y-3x^2y}{1-4xy+x^2y^2-y-x^2y} \\ &= 2n + n[x^ny^n]\frac{1-x^2y^2}{1-4xy+x^2y^2-y-x^2y} \end{aligned}$$

since the constant term in y is irrelevant. Letting $z = xy$, we now have

$$\begin{aligned} c_n &= 2n + n[z^n]\frac{1-z^2}{(1-z)^2-y(1+x)^2} \\ &= 2n + [z^n]z\frac{\partial}{\partial z}\frac{1-z^2}{(1-z)^2}\left(1-y\frac{(1+x)^2}{(1-z)^2}\right)^{-1} \\ &= 2n + [z^n]z\frac{\partial}{\partial z}\frac{1+z}{1-z}\sum_{i\geq 0}\frac{y^i(1+x)^{2i}}{(1-z)^{2i}} \\ &= 2n + [z^n]z\frac{\partial}{\partial z}\frac{1+z}{1-z}\sum_{i\geq 0}\binom{2i}{i}\left(\frac{z}{(1-z)^2}\right)^i \end{aligned}$$

upon extracting $[x^i](1+x)^{2i}$, and using the fact that $\sum_{i\geq 0}\binom{2i}{i}t^i = (1-4t)^{-1/2}$, we obtain

$$\begin{aligned} c_n &= 2n + [z^n]z\frac{\partial}{\partial z}\frac{1+z}{1-z}\left(1-\frac{4z}{(1-z)^2}\right)^{-1/2} \\ &= 2n + [z^n]z\frac{\partial}{\partial z}\frac{1+z}{\sqrt{(1-z)^2-4z}} = 2n + [z^n]z\frac{\partial}{\partial z}(1+z)a^{-1/2} \end{aligned}$$

where $a = 1-6z+z^2$. Finally, we simplify to obtain

$$\begin{aligned} c_n &= [z^n]\left\{\frac{2z}{(1-z)^2} + z(a^{-1/2} - \tfrac{1}{2}(1+z)a^{-3/2}(-6+2z))\right\} \\ &= [z^n]\left\{\frac{2z}{(1-z)^2} + \frac{4z(1-z)}{a^{3/2}}\right\} = [z^n]\frac{2z(a^{3/2}+2(1-z)^3)}{a^{3/2}(1-z)^2}, \end{aligned}$$

agreeing with [1, p. 141]

4.2 Zippers

Those not familiar with generating functions may prefer enumerations where the result is expressed in terms of sums of products of choice functions. The results of [6, 7, 3] are given in this form. Of course, such an expression is easily obtained from a generating function. In the case of ordinary crowns, our previous example, it was extra work to derive a generating function. In the present section we will leave our result in terms of choice functions.

The **zipper** Z_n on $2n$ elements is obtained by identifying elements of $\mathcal{B}_{n+1}$ which are congruent modulo $2n$ (See Figure 3). Let z_n be the number of endomorphisms of a zipper on $2n$ elements. We get

$$A = \begin{bmatrix} x^{-2}+1+x^2 & x^{-2}+1 & 1+x^2 & 1 & x \\ 1+x^2 & 1 & 1+x^2+x^4 & 1 & x^3 \\ x^{-2}+1 & x^{-4}+x^{-2}+1 & 1 & 1 & x^{-1} \\ 1 & 1 & 1 & 1 & 0 \\ x^{-1} & x^{-3} & x & 0 & 1 \end{bmatrix}$$

and $z_n = n[(x^{-2n}+x^0+x^{2n})y^n]\text{trace}\,(I-yA)^{-1}$. If x is replaced by x^{-1} in A, we obtain the transpose of A, so $[x^{-2n}y^n]\text{trace}\,(I-yA)^{-1} = [x^{2n}y^n]\text{trace}\,(I-yA)^{-1}$. Using this observation and multiplying the entries of A by x^2 (in order to obtain a formal power series), we have

$$z_n = n\{[x^{2n}] + 2[x^0]\}[y^n]\text{trace}\,(I - yx^2A)^{-1}.$$

We can obtain an expression for trace $(I - yx^2A)^{-1}$. It is an even expression in x and if we perform a partial fraction expansion and ignore the constant term in y we obtain

$$z_n = n\{[x^n] + 2[x^0]\}[y^n](A + 2B)$$

where

$$\begin{aligned} A &= \frac{1+2x^2y^2}{1+y-xy+x^2y-2x^2y^2}, \\ B &= \frac{1-y-2xy-x^2y}{1-2y-4xy-2x^2y+y^2+2xy^2-x^2y^2+2x^3y^2+x^4y^2}. \end{aligned}$$

It is straightforward to check that $[x^0y^n]A = (-1)^n$ and $[x^0y^n]B = 1$. Letting $z = xy$, we now have

$$[x^ny^n]A = [x^ny^n]\frac{1+2z^2}{1+z-2z^2+y(1-x)^2}$$

$$
\begin{aligned}
&= [z^n]\frac{1+2z^2}{1+z-2z^2}\left(1+\frac{y(1-x)^2}{1+z-2z^2}\right)^{-1} \\
&= [z^n](1+2z^2)\sum_{i\geq 0}(-1)^i\frac{y^i(1-x)^{2i}}{(1+z-2z^2)^{i+1}} \\
&= [z^n](1+2z^2)\sum_{i\geq 0}\binom{2i}{i}z^i(1-z)^{-(i+1)}(1+2z)^{-(i+1)}
\end{aligned}
$$

upon extracting $[x^i](1+x)^{2i}$. Using similar techniques and letting $w = y(1+x)^2$, we can also show that

$$
\begin{aligned}
[x^n y^n]B &= [z^n]\frac{1-w}{1-3z^2-2w+w^2-2zw} \\
&= [z^n]\frac{1}{1-w}\left(1-\frac{z(2w+3z)}{(1-w)^2}\right)^{-1} \\
&= [z^n]\sum_{i\geq 0}\frac{z^i(2w+3z)^i}{(1-w)^{2i+1}}.
\end{aligned}
$$

Extracting the above coefficients and putting the terms together, we conclude that

$$
\begin{aligned}
z_n = n\Big\{4+2(-1)^n &+ \sum_{i,j}\binom{2i}{i}\binom{i+j}{j}\left\{\binom{n-j}{i}+2\binom{n-j-2}{i}\right\}(-2)^j \\
&+ \sum_{i,j}\binom{i}{j}\binom{n}{2i}\binom{2n-4i+2j}{n-2i+j}2^{j+1}3^{i-j}\Big\}.
\end{aligned}
$$

5 Concluding Remarks and Numerical Data

There is another way to exploit the equation $\phi_{G_n} = \text{trace}(A^n)$ that deserves mention. It may be that the matrix A is diagonalizable, so that we can write $A = SDS^{-1}$, D diagonal. This is the case, for example, with ordinary crowns. In this case, we have

$$\phi_{G_n} = \text{trace}(SD^nS^{-1}).$$

Since D is diagonal, we can get a closed form for D^n, and hence ϕ_{G_n} as a function of n. In any case, the current formulation $\phi_{G_n} = \text{trace}(A^n)$ allows easy computation of the number of maps for specific n for the generalized crowns corresponding to any of the examples in Figure 3. For $n \geq 2$, let D_n be the generalized crown

Table 1: Numbers of endomorphisms of Z_n and D_n.

n	z_n	d_n
2	275	139
3	951	1,646
4	4,868	22,075
5	31,735	310,442
6	252,054	4,471,966
7	1,980,727	65,398,070
8	15,463,416	966,609,787
9	119,914,191	14,401,689,461
10	924,752,690	215,922,873,094
11	7,097,502,159	3,253,709,282,423
12	54,253,458,780	49,234,244,569,030
13	413,281,739,949	747,605,163,039,752
14	3,138,868,642,826	11,385,905,901,377,440

corresponding to $\mathcal{D}_n$. Thus D_n has the six element base unit $1 < 2 < 3 > 4 < 5 < 6$ as in Figure 3. Let d_n be the number of endomorphisms of D_n. We give values for z_n and d_n in Table 1 .

References

[1] J.D. Currie and T.I. Visentin (1991) The number of order-preserving maps of fences and crowns, *Order* **8**, 133–142.

[2] D. Duffus, V. Rodl, B. Sands and R. E. Woodrow (1992) Enumeration of order preserving maps *Order* **9**, 15–29.

[3] J. D. Farley (1995) The number of order-preserving maps between fences and crowns, *Order* **12**, 5–44.

[4] K. Parol and A. Rutkowski (1993) Counting the number of isotone selfmappings of crowns (1993) *Order* **10**, 221–226.

[5] I. Rival and A. Rutkowski (1991) Does almost every isotone self-map have a fixed point? In *Extremal Problems for Finite Sets, Bolyai Soc. Math. Studies*, **3**, Visegrád, Hungary, 413–422.

[6] A. Rutkowski (1992) The number of strictly increasing mappings of fences. *Order* **9**, 31–42.

[7] A. Rutkowski (1992) The formula for the number of order preserving self-mappings of a fence, *Order* **9**, 127–137.

[8] N. Zaguia (1993) Isotone maps: enumeration and structure, In *Finite and Infinite Combinatorics in Sets and Logic*, N. W. Sauer, R. E. Woodrow and B. Sands (eds), Kluwer Academic Publishers, Dordrecht, 421–430.